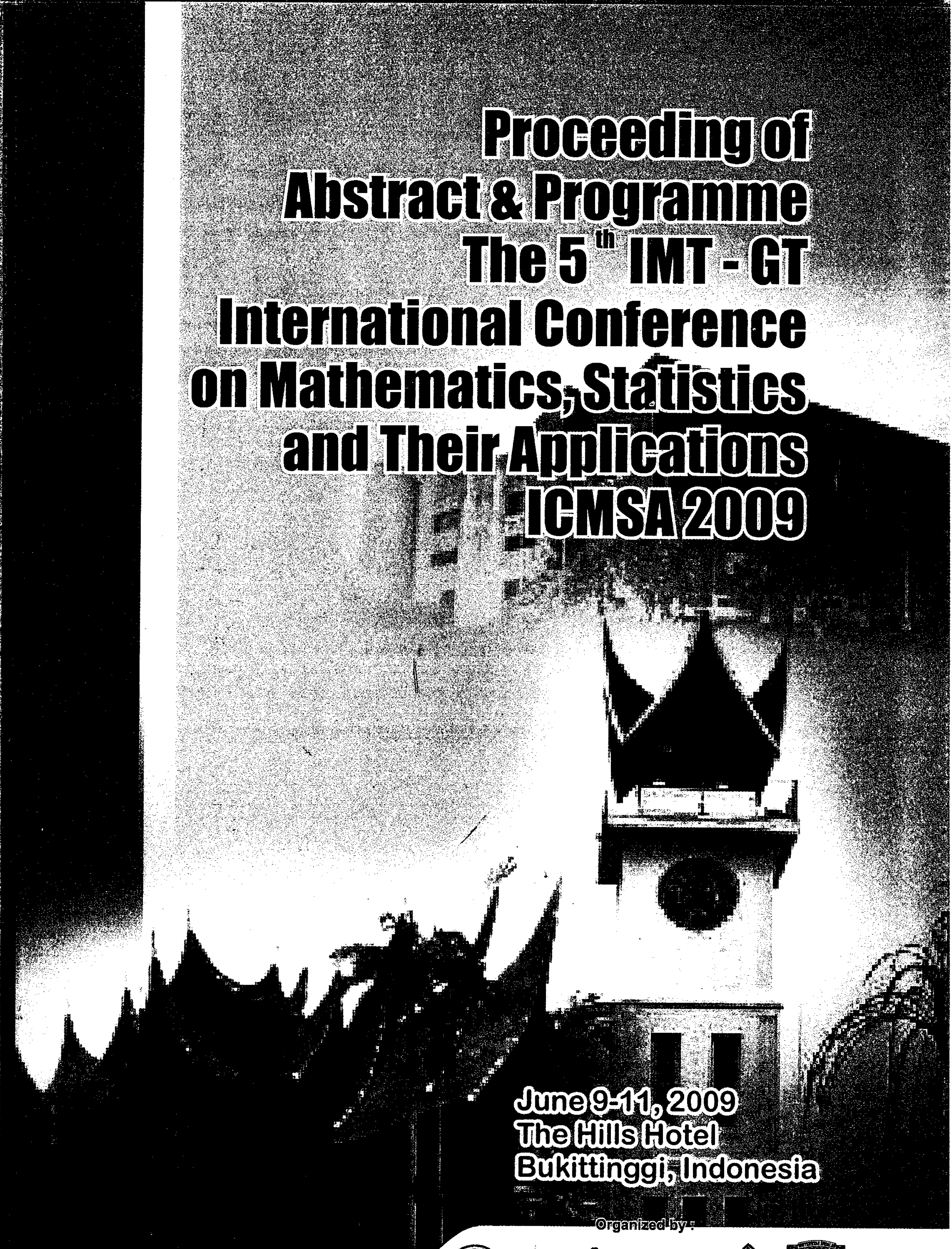

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Simple Version of the Linear Business Cycle Model

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ABSTRACT

The paper developed a linear business cycle model. To analyze the impact of small changes in productivity, the model is linearized around the steady state which allows us to find optimal level of consumption and investment under assumption of perfect foresight i.e, knowledge of all future values of the total factor productivity.

Keywords: Cobb-Douglas Production function, Dynamics modeling, Linear Business Cycle, Neoclassical Model, Utility function.

1. Introduction

The aim of the paper is to develop a simple version of the neoclassical model with an infinitely lived representative household and labor-augmenting technological progress. When shocks to productivity are random, the precise future values of the total factor productivity are random, the precise future values of the total factor productivity are no longer known. However, knowledge of the stochastic process that governs the changes in productivity allows us to base our decisions on estimated future values. So once the process for the total factor productivity is specified, the simulation of all macroeconomic time series is straightforward.

2. The Basic Neoclassical Model

Preferences of an infinitely lived individual are represented by the utility function

$$U=\sum_{t=0}^{\infty}\beta^{t}u\left(C_{t},L_{t}\right) \tag{1}$$

Where u has constant intertemporal elasticity of substitution with respect to consumption and leisure

$$u\left(C_{t},L_{t}\right)=\frac{1}{1-\sigma}C_{t}^{1-\sigma}v\left(L_{t}\right) \tag{2}$$

C_t and L_t are consumption and leisure in period t.

The only goods in the economy is produced according to the technology

$$Y_t = A_t K_t^{1-\alpha} \left(N_t X_t \right)^{\alpha} \tag{3}$$

where K = Capital

N = Time devoted to work

X = Growth rate of labor augmenting technical progress.

Y = output

Labor augmenting technology is used to achieve a steady state, in which the steady state growth of the labor force is zero.

Goods in each period can be either invested or consumed. Capital depreciates at rate δ.

$$K_{t+1} = I_t + (1-\delta) K_t \tag{4}$$

There are two resource constraints in each period. First, time devoted to work and leisure must be equal to the time endowment. Second, consumption and investment must not exceed output.

$$L_t + N_t = 1 \tag{5}$$

$$C_t + I_t \le Y_t \tag{6}$$

To obtain an economy with steady state, all variables must be converted into per efficiency units so that $c = \dfrac{C}{X}, k = \dfrac{K}{X}, i = \dfrac{I}{X}$ etc. Utility function, technology and resource constraints are then expressed in per efficiency variables. Assuming a constant growth rate of the labor augmenting technical progress, that is

$$X_t = e^{(\gamma-1)t} \tag{7}$$

We obtain

$$U = \sum_{t=0}^{\infty} (\beta^*)^t \frac{1}{1-\sigma} c_t^{1-\sigma} v(L_t) \tag{8}$$

$$y_t = A_t k_t^{1-\alpha} N^{\alpha} \tag{9}$$

$$\gamma^{k_{t+1}} = i_t + (1-\delta) k_t \tag{10}$$

$$y_t \ge i_t + c_t \tag{11}$$

where $\beta^* = \beta\gamma^{1-\sigma}$. This economy is solved with the assumption of perfect foresight. The optimal path of capital and consumption implies

$$D_1 u(c_t, L_t) - \lambda_t = 0 \tag{12}$$

$$D_2 u(c_t, L_t) - \lambda_t A_t D_2 F(k_t, N_t) = 0 \tag{13}$$

$$\beta^* \lambda_{t+1} \left[A_{t+1} D_1 F(k_{t+1}, N_{t+1}) + (1-\delta) \right] - \lambda_t \gamma = 0 \tag{14}$$

$$A_t F(k_t, N_t) + (1-\delta) k_t - \gamma k_{t+1} - c_t = 0 \tag{15}$$

$$\lim_{t \to 0} (\beta^*)^t \lambda_t k_{t+1} = 0 \tag{16}$$

To obtain a quantitative analysis of the impact of changes in A_t, equations (12) – (15) are linearized around the steady state (*A, k, N, c, y*). All variables are expressed in their percentage deviations from the steady state

$$\hat{c}_t = \ln\frac{c_t}{c} = \frac{c_t - c}{c} \tag{17}$$

The linearization proceeds as follows: to approximate function *f*(*x, y, …*) around $(x_0, y_0, \ldots)$ one uses the Taylor series

$$\begin{aligned} f(x,y,\ldots) &= \frac{\partial f(x_0,y_0,\ldots)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0,\ldots)}{\partial y}(y-y_0)+\ldots \\ &= x_0\frac{\partial f(x_0,y_0,\ldots)}{\partial x}\frac{(x-x_0)}{x_0} + y_0\frac{\partial f(x_0,y_0,\ldots)}{\partial y}\frac{(y-y_0)}{y_0}+\ldots \\ &= x_0\frac{\partial f(x_0,y_0,\ldots)}{\partial x}\hat{x} + y_0\frac{\partial f(x_0,y_0,\ldots)}{\partial y}\hat{y}+\ldots = \\ &= f(x_0,y_0,\ldots)\left(\frac{\partial \ln f(x_0,y_0,\ldots)}{\partial \ln x}\hat{x} + \frac{\partial \ln f(x_0,y_0,\ldots)}{\partial \ln y}\hat{y}+\ldots\right) \end{aligned}$$

In this manner equations (12) – (15) imply

$$\xi_{cc}\hat{c}_t - \xi_{cl}\frac{N}{1-N}\hat{N}_t - \hat{\lambda}_t = 0 \tag{18}$$

$$\xi_{lc}\hat{c}_t - \xi_{ll}\frac{N}{1-N}\hat{N}_t - \hat{\lambda}_t - \hat{A}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t = 0 \tag{19}$$

$$\hat{\lambda}_{t+1} + \eta_A\hat{A}_{t+1} + \eta_k\hat{k}_{t+1} + \eta_N\hat{N}_{t+1} = \hat{\lambda}_t \tag{20}$$

$$\begin{aligned} \hat{y}_t &= \hat{A}_t + \alpha\hat{N}_t + (1-\alpha)\hat{k}_t \\ &= s_c\hat{c}_t + s_i\Phi\hat{k}_{t+1} - s_i(\Phi-1)\hat{k}_t \end{aligned} \tag{21}$$

where $\xi_{ab} = \dfrac{d\ln u_a}{d\ln b}$ is the elasticity of marginal utility of *a* with respect to *b*. For the Cobb-Douglas technology $\eta_A = \dfrac{\gamma - \beta(1-\delta)}{\gamma}, \eta_k = -\alpha\eta_A, \eta_N = \alpha\eta_A$, s_i and s_c are shares of investment and consumption in output $s_c + s_i = 1, s_i = (1-\alpha)\dfrac{\gamma+\delta-1}{r+\delta}$.

To find the dynamic of the capital formation one must solve a system of two first order difference equations in $\hat{\lambda}_t$ and $\hat{k}_t$ which are obtained from equations ((18), (19), (21)

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{pmatrix} = W \begin{pmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{pmatrix} + R\hat{A}_{t+1} + Q\hat{A}_t \tag{22}$$

The solution to a general problem

$$x_{t+1} = Wx_t + v_t \tag{23}$$

with given x_0 and v_t can be expressed in the form

$$x_t = W^t x_0 - \sum_{j=0}^{\infty} W^{-j-1} v_{t+j} \tag{24}$$

To satisfy the transversality condition we choose λ_0 such that $Wx_0 = \mu_1 x_0$, where μ_1 is the smaller of the two eigenvalues of *W*. Equation (24) can then be rewritten as

$$x_t = \mu_1 x_0 - \sum_{j=0}^{\infty} W^{-j-1} v_{t+j} \tag{25}$$

Equation (25) allows us to express x_{t+1} in the form

$$\begin{aligned} x_{t+1} &= \mu_1 x_0 - \sum_{j=0}^{\infty} W^{-j-1} v_{t+1+j} + \mu_1 \sum_{j=0}^{\infty} W^{-j-1} v_{i+j} \\ &= \mu_1 x_t + \mu_1 W^{-1} v_t + \sum_{j=0}^{\infty} \left(\mu_1 W^{-1} - I \right) v_{t+1+j} \\ &= \mu_1 x_1 + \mu_1 W^{-1} v_t + P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} \left(\frac{\mu_1}{\mu_2} - 1 \right) \sum_{j=0}^{\infty} \mu_2^{-j-1} v_{t+1+j} \\ &= \mu_1 x_t + B_1 v_t + B_2 \sum_{j=0}^{\infty} \mu_2^{-j-1} v_{t+1+j} \end{aligned} \tag{26}$$

Columns of matrix **P** are formed by eigenvectors of matrix **W**

$$W = P \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} P^{-1} \tag{27}$$

Solution (26) allows us to decide on the capital stock of next period based on the capital stock of this period plus present and future values of the total factor productivity $\hat{A}_t$.

$$\hat{k}_{t+1} = \mu_1 \hat{k}_t + \psi_1 \hat{A}_t + \psi_2 \sum_{j=0}^{\infty} \mu_2^{-j} \hat{A}_{t+1+j} \tag{28}$$

Expressions for ψ_1, ψ_2 can be found by comparing equations (26) and (28).

3. Real Business Cycles

The real business cycle model differs from the basic neoclassical model in that we remove the assumption of perfect foresight and introduce a stochastic behavior to the total factor productivity.

We start with equation (28) describing the path of capital formation. Then we specify a particular stochastic process for $\hat{A}_t$, for example in case AR(1) with parameter ρ, and replace $\hat{A}_{t+j}$ with their expected values given information at time t. The dynamics of the state variables $\hat{k}_t, \hat{A}_t$ is then given by the linear system

$$s_{t+1} = \begin{pmatrix} \hat{k}_{t+1} \\ \hat{A}_{t+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & \pi_{kA} \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix} + \begin{pmatrix} 0 \\ \in_{A,t+1} \end{pmatrix} \tag{29}$$

$$= Ms_t + \in_{t+1}$$

Additional linear equations specify how consumption, work effort, investment and output depend on the state variables s_t. These equations are derived from equations (18) – (21) with the use of equation (29). In the vector notation they take the form

$$z_t = \begin{pmatrix} \hat{c}_t \\ \hat{N}_t \\ \hat{y}_t \\ \hat{i}_t \end{pmatrix} = \begin{pmatrix} \pi_{ck} & \pi_{cA} \\ \pi_{Nk} & \pi_{NA} \\ \pi_{yk} & \pi_{yA} \\ \pi_{ik} & \pi_{iA} \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix} = \prod s_t \tag{30}$$

The preceding formulation facilitates calculation of population moments knowing variance-covariance matrix Σ_{tt} of the state vector s_t.

$$\operatorname{var}\left(x_{t+j} x_t^T\right) = M^j \Sigma_{tt} \tag{31}$$

$$\operatorname{var}\left(z_{t+j} x_t^T\right) = \prod M^j \Sigma_{tt} \tag{32}$$

It can easily be verified that

$$\Sigma_{11} = \operatorname{var}\left(\hat{A}_t\right) = \frac{\sigma^2}{1-\rho^2} \tag{33}$$

$$\Sigma_{22} = \operatorname{var}\left(\hat{k}_t\right) = \frac{\pi_{kA}^2 \sigma^2}{(\rho-\mu_1)^2} \left(\frac{\rho^2}{1-\rho^2} + \frac{\mu_1^2}{1-\mu_1^2} - \frac{2\mu_1\rho}{1-\mu_1\rho} \right) \tag{34}$$

$$\Sigma_{12} = \Sigma_{21} = \operatorname{cov}(\hat{A}_t, \hat{k}_t) = \frac{\pi_{kA}\sigma^2}{\rho-\mu_1} \left(\frac{\rho}{1-\rho^2} - \frac{\mu_1}{1-\rho\mu_1} \right) \tag{35}$$

4. Conclusion

The timing of the series will bring ambiguous results. On the other hand, the model can predict that the total factor productivity is coincidental with output. The model is capable of matching volatility and procyclicality of all variables

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