MONOPOLE-ANTIMONOPOLE AND VORTEX RINGS*

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Abstract

The SU(2) Yang-Mills-Higgs theory supports the existence of monopoles, antimonopoles, and vortex rings. In this paper, we would like to present new exact static antimonopole-monopole-antimonopole (A-M-A) configurations. The net magnetic charge of these configurations is always negative one, whilst the net magnetic charge at the origin is always positive one for all positive integer values of the solution parameter m. However, when m increases beyond one, vortex rings appear coexisting with these A-M-A configurations. The number of vortex rings increases proportionally with the value of m. They are magnetically neutral and are located in space where the Higgs field vanishes. We also show that a single point singularity in the Higgs field need not corresponds to a structureless 1-monopole at the origin but to a zero size monopole-antimonopole-monopole (MAM) structure. These exact solutions are a different kind of BPS solutions as they satisfy the first order Bogomol'nyi equation but possess infinite energy due to a point singularity at the origin of the coordinate axes. They are all axially symmetrical about the z-axis.

1 Introduction

The SU(2) Yang-Mills-Higgs (YMH) field theory, with the Higgs field in the adjoint representation possesses magnetic monopole, multimonopole, antimonopoles, and vortex rings solutions [1]-[8]. The only spherically symmetric monopole solution is the unit magnetic charge monopole. The 't Hooft-Polyakov monopole solution with non zero Higgs mass and Higgs self-interaction is numerically, spherically symmetrical [1]-[2]. Multimonopole solutions possess at most axial symmetry [3].

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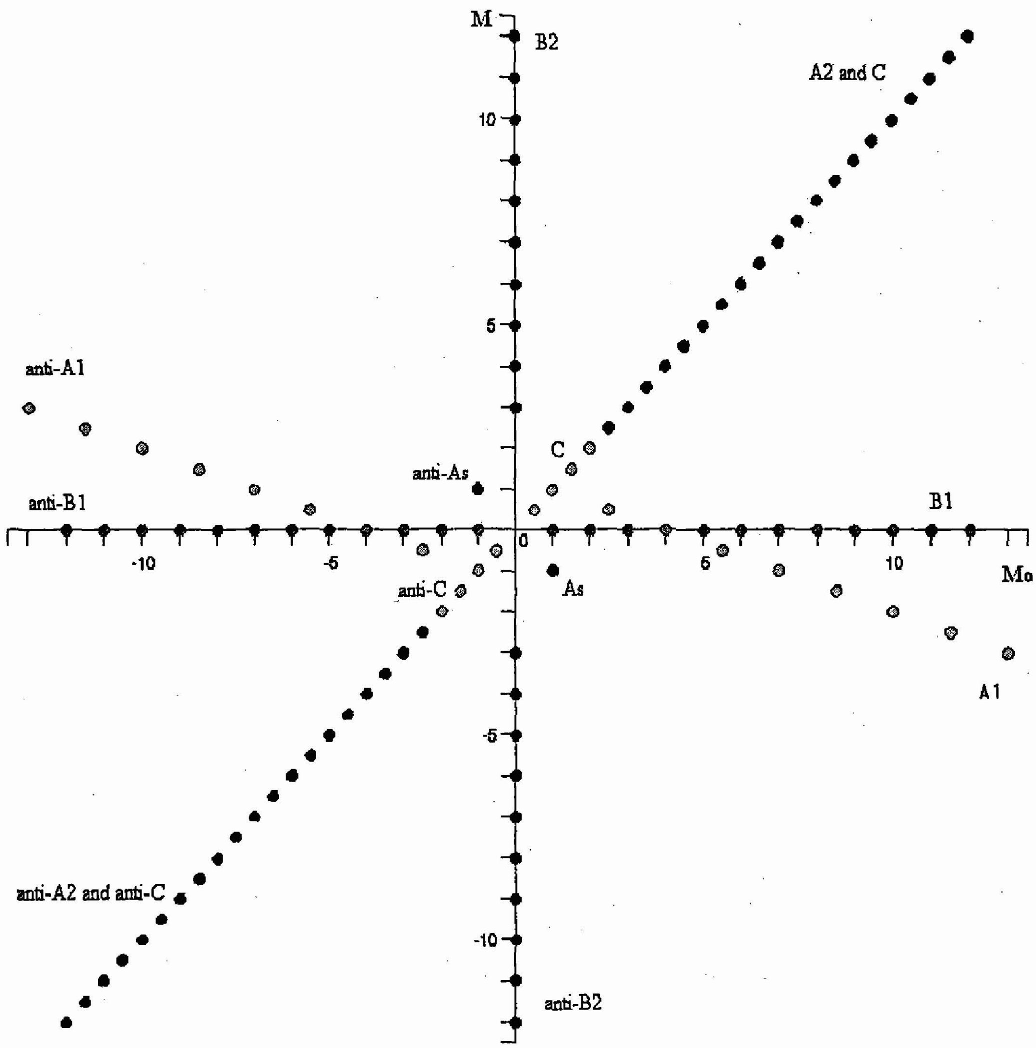

Figure 4: A point plot of the net magnetic charges M at large r versus the magnetic charges M_0 at $r = 0$ for the series of configurations and anti-configurations of A1, A2, B1, B2, C and AS.

The model with non vanishing Higgs vacuum expectation value but vanishing Higgs potential possesses exact monopole and multimonopole solutions which can be obtained by solving the first order Bogomol'nyi equations [11]. These solutions satisfying the Bogomol'nyi-Prasad-Sommerfield (BPS) limit possess minimal energies. Exact monopole and multimonopoles solutions exist in the BPS limit [2] - [3] whilst outside the BPS limit, when the Higgs field potential is non vanishing only numerical solutions are known. Asymmetric multimonopole solutions with no rotational symmetry are numerical solutions even in the BPS limit [4].

At present, the different exact configurations of monopoles found are the BPS multimonopole solutions of magnetic charges M greater than unity with all the magnetic charges superimposed into a single point in space [3]. These superimposed multimonopole solutions possess axial and mirror symmetries. Following these works, finitely separated 1-monopoles were also constructed [4]-[5].

Numerical axially symmetric non-Bogomol'nyi monopole-antimonopole chain solutions were also found to exist both in the limit of a vanishing Higgs potential as well as in the presence of a finite Higgs potential. Recently, numerical BPS axially symmetric vortex rings solutions have also been reported [6].

We have reported on the existence of a different type of BPS static monopole-antimonopole solution in Ref. [7]. This solution which is exact and axially symmetric, represents two separate antimonopoles located at equal distances along the z-axis from a 1-monopole which is located at the origin. We have also shown that the extended ansatz of Ref. [7] possesses more multimonopole-antimonopole configurations, together with their anti-configurations [8]. These configurations possess either radial, axial, or mirror symmetries about the z-axis and they represent different combinations of monopoles, multimonopole, and antimonopoles.

In general, configurations of the YMH field theory with a unit magnetic charge are spherically symmetric [1], [2], whilst multimonopole configurations with magnetic charges greater than unity possess at most axial symmetry [3]. However we have emphasized in a recent work [9] that unit magnetic charge configurations are not necessarily spherically symmetric. By employing the ansatz of Ref.[7] we have found exact unit magnetic charge solution that does not even possess axial symmetry but only mirror symmetry about a vertical plane through the z-axis. However the reverse is true and it has been shown that multimonopole solutions cannot possess spherical symmetry [10]. We would also like to mention that within the ansatz of Ref. [7], half-monopole solutions have also been reported [9].

In this paper we would like to present new static axially symmetric antimonopole-monopole-antimonopole (A-M-A) configurations of the SU(2) YMH theory with the Higgs field in the adjoint representation. Here the Higgs field vanishes either at points corresponding to antimonopoles or at rings corresponding to vortex loops. The net magnetic charge of these configurations is always negative one, whilst the net magnetic charge at the origin, $r = 0$, is always positive one for all positive integer values of the solution parameter m. However, when m increases beyond one, vortex rings appear coexisting with these A-M-A configurations. The number of vortex rings in the configuration is equal to $(m - 1)$ where $m \geq 1$. They are magnetically neutral and are located horizontally in space where the Higgs field is zero. Hence this family of solutions all lies in the topologically non trival sector with topogical charge negative one.

The two antimonopoles of the solutions are located at the two zeros of the Higgs field along the z-axis, whilst the 1-monopole is located at a point singularity of the Higgs field at the origin. . We also show that this single point singularity in the Higgs field need not corresponds to a structureless 1-monopole at the origin but to a zero size monopole-antimonopole-monopole (MAM) structure when $m = 1$. These exact solutions are a different kind of BPS solutions as they satisfy the first order Bogomol'nyi equation but possess infinite energy due to a point singularity at the origin of the coordinate axes.

We briefly review the SU(2) Yang-Mills-Higgs field theory and discussed the boundary conditions of these solutions in the next section. We present the solutions and discuss the t' Hooft Abelian magnetic charges and fields in section 3. We end with some comments in section 4.

2 The SU(2) Yang-Mills-Higgs Theory

The SU(2) group admits the triplet Yang-Mills gauge fields potential A^a_μ which when coupled to a scalar Higgs triplets field Φ^a in 3+1 dimensions gives the SU(2) YMH theory [12]. The index a is the SU(2) internal space index and for a given a, Φ^a is a scalar whereas A^a_μ is a vector under Lorentz transformation. The SU(2) YMH Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2, \quad (1)$$

where the Higgs field mass, μ, and the strength of the Higgs potential, λ, are constants. The vacuum expectation value of the Higgs field is then $\mu/\sqrt{\lambda}$. The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$\begin{aligned} D_\mu\Phi^a &= \partial_\mu\Phi^a + \epsilon^{abc}A^b_\mu\Phi^c, & (2)\\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu. & (3) \end{aligned}$$

Since the gauge field coupling constant g can be scaled away, we can set g to one without any loss of generality. The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) internal group indices a, b, c run from 1 to 3 and the spatial indices are $\mu, \nu, \alpha = 0, 1, 2$, and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$\begin{aligned} D^\mu F^a_{\mu\nu} &= \partial^\mu F^a_{\mu\nu} + \epsilon^{abc}A^{b\mu}F^c_{\mu\nu} = \epsilon^{abc}\Phi^b D_\nu\Phi^c, \\ D^\mu D_\mu\Phi^a &= -\lambda\Phi^a(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}). & (4) \end{aligned}$$

The tensor to be identified with the Abelian electromagnetic field, as proposed by 't Hooft [1], [13] is

$$\begin{aligned} F_{\mu\nu} &= \hat{\Phi}^a F^a_{\mu\nu} - \epsilon^{abc}\hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc}\hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c, \end{aligned} \tag{5}$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, $\hat{\Phi}^a = \Phi^a/|\Phi|$, $|\Phi| = \sqrt{\Phi^a\Phi^a}$. Hence the Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, where the indices, $i, j, k = 1, 2, 3$. The topological magnetic current, which is also the topological current density of the system is [13]

$$k_\mu = \frac{1}{8\pi}\ \epsilon_{\mu\nu\rho\sigma}\ \epsilon_{abc}\ \partial^\nu \hat{\Phi}^a\ \partial^\rho \hat{\Phi}^b\ \partial^\sigma \hat{\Phi}^c. \tag{6}$$

Therefore the corresponding conserved topological magnetic charge is

$$\begin{aligned} M &= \int d^3x\ k_0 = \frac{1}{8\pi}\int \epsilon_{ijk}\epsilon^{abc}\partial_i\left(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) d^3x \\ &= \frac{1}{8\pi}\oint d^2\sigma_i \left(\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) \\ &= \frac{1}{4\pi}\oint d^2\sigma_i\ B_i. \end{aligned} \tag{7}$$

Our work is restricted to the static case where $A^a_0 = 0$ with massless Higgs field and vanishing self-interaction. The magnitude of the Higgs field vanishes as $1/r$ at large r. However, this does not affect the Abelian magnetic field of the solutions as this magnetic field depends only on the unit vector of the Higgs field. It is in this limit that the solutions are solved using both the second order Euler-Lagrange equations and the Bogomol'nyi equations, $B^a_i \pm D_i\Phi^a = 0$. The $\pm$ sign corresponds to monopoles and antimonopoles respectively for the usual BPS solutions [13]. In our case, the A-M-A configuration is solved with the $+$ sign and its anti-configuration, that is, the M-A-M configuration is solved with the $-$ sign [8].

3 Monopole, Antimonopoles, and Vortex Rings

The static gauge fields and Higgs field which will lead to the axially symmetric vortex rings solutions are given respectively by [7]

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\psi(r)\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}R(\theta)\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \Phi_1\ \hat{r}^a + \Phi_2\ \hat{\theta}^a, \end{aligned} \tag{8}$$

where $\Phi_1 = \frac{1}{r}\psi(r)$, $\Phi_2 = \frac{1}{r}R(\theta)$. The spherical coordinate orthonormal unit vectors, $\hat{r}^a$, $\hat{\theta}^a$, and $\hat{\phi}^a$ are defined by

$$\begin{aligned} \hat{r}^a &= \sin\theta\ \cos\phi\ \delta^a_1 + \sin\theta\ \sin\phi\ \delta^a_2 + \cos\theta\delta^a_3, \\ \hat{\theta}^a &= \cos\theta\ \cos\phi\ \delta^a_1 + \cos\theta\ \sin\phi\ \delta^a_2 - \sin\theta\ \delta^a_3, \\ \hat{\phi}^a &= -\sin\phi\ \delta^a_1 + \cos\phi\ \delta^a_2, \end{aligned} \tag{9}$$

where $r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3/r)$, and $\phi = \tan^{-1}(x_2/x_1)$. The gauge field strength tensor and the covariant derivative of the Higgs field are given respectively by

$$\begin{aligned}
F^a_{\mu\nu} &= \frac{1}{r^2}\hat{r}^a\left\{\dot{R} + R\cot\theta + 2\psi - \psi^2\right\}(\hat{\phi}_\mu\hat{\theta}_\nu - \hat{\phi}_\nu\hat{\theta}_\mu) \\
&+ \frac{1}{r^2}\hat{\theta}^a\left\{R(1-\psi)\right\}(\hat{\phi}_\mu\hat{\theta}_\nu - \hat{\phi}_\nu\hat{\theta}_\mu) \\
&+ \frac{1}{r^2}\left\{\hat{r}^a R(1-\psi) + \hat{\theta}^a(r\psi' + R\cot\theta - R^2)\right\}(\hat{r}_\mu\hat{\phi}_\nu - \hat{r}_\nu\hat{\phi}_\mu) \\
&+ \frac{1}{r^2}\hat{\phi}^a\left\{-(r\psi' + \dot{R})\right\}(\hat{r}_\mu\hat{\theta}_\nu - \hat{r}_\nu\hat{\theta}_\mu),
\end{aligned} \tag{10}$$

$$\begin{aligned}
D_\mu\Phi^a &= \frac{1}{r^2}\left\{\hat{r}^a(r\psi' - \psi - R^2) - \hat{\theta}^a R(1-\psi)\right\}\hat{r}_\mu \\
&+ \frac{1}{r^2}\left\{-\hat{r}^a R(1-\psi) + \hat{\theta}^a(\dot{R} + \psi - \psi^2)\right\}\hat{\theta}_\mu \\
&+ \frac{1}{r^2}\left\{\hat{\phi}^a(\psi - \psi^2 + R\cot\theta - R^2)\right\}\hat{\phi}_\mu.
\end{aligned} \tag{11}$$

Here prime means $\partial/\partial r$ and dot means $\partial/\partial\theta$. The gauge fixing condition that we used here is the radiation or Coulomb gauge, $\partial^i A^a_i = 0$, $A^a_0 = 0$.

The ansatz (8) is substituted into the equations of motion (4) as well as the Bogomol'nyi equations with the positive sign and the resulting equations of motion are just two first order differential equations,

$$r\psi' + \psi - \psi^2 = -p, \tag{12}$$

$$\dot{R} + R\cot\theta - R^2 = p, \tag{13}$$

where p is an arbitary constant. Eq.(12) is exactly solvable for all real values of p and the integration constant can be scaled away by letting $r \to r/c$, where c is the arbitrary integration constant. In order to obtain solutions of ψ with $(2m+1)$ powers of r we can write $p = m(m+1)$ where m is real. By doing so, the solutions of the Riccati equation (13) can be exactly solved in terms of the Legendre functions of the first and second kind. For the solutions of Eq.(13) to be regular along the z-axis, we require $R(\theta)$ to vanish when $\theta = 0$ and $\theta = \pi$. To achieve these boundary conditions, the integration constant of Eq.(13) is set to zero and m is restricted to take integer values. The solutions for ψ and R are then given respectively by

$$\begin{aligned}
\psi(r) &= \frac{(m+1) - mr^{2m+1}}{1 + r^{2m+1}}, \\
R(\theta) &= (m+1)\left\{\cot\theta - \frac{P_{m+1}(\cos\theta)}{P_m(\cos\theta)}\csc\theta\right\},
\end{aligned} \tag{14}$$

where P_m is the Legendre polynomial of degree m, and $m = 0, 1, 2, 3, \ldots$. Hence the boundary conditions of the solutions, Eq.(14), are $\psi(0) = m + 1$, $\psi(\infty) = -m$, $R(0) = R(\pi) = 0$.

In the BPS limit, the energy can be written in the form

$$\begin{aligned} E &= \pm \int \partial_i (B_i^a \Phi^a) \, d^3x + \int \frac{1}{2} (B_i^a \pm D_i \Phi^a)^2 \, d^3x \\ &= \pm \int \partial_i (B_i^a \Phi^a) \, d^3x = 4\pi M \frac{\mu}{\sqrt{\lambda}}, \end{aligned} \quad (15)$$

where M is the "topological charge" when the vacuum expectation value of the Higgs field, $\frac{\mu}{\sqrt{\lambda}}$ is non zero coupled with some non-trivial topological structure of the fields at large r. The energy of the systems here is however not finite for all values of m at the origin $r = 0$ due to the presence of a singularity there. Also the vacuum expectation values of our solutions tend to zero at large r. Hence the energy of our solutions is not bounded from below by $4\pi M \frac{\mu}{\sqrt{\lambda}}$.

However the topological charge is also related to another gauge invariant quantity of the system as given by Eq.(7) which corresponds to the zeros of the Higgs field,

$$M_\infty = \frac{1}{8\pi} \oint d^2\sigma_i \left(\epsilon_{ijk} \epsilon^{abc} \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \right) \Big|_{r \to \infty}. \quad (16)$$

The monopoles and antimonopoles of our solutions here can then be associated with the number of zeros of Φ^a enclosed by the sphere at infinity. The positions of the antimonopole do correspond to the zeros of the Higgs field in the A-M-A solutions. However the 1-monopole is not located at the zeros of the Higgs field but at the origin of the coordinate axes where the Higgs field is singular. The reverse is true of the case of its anti-configuration.

From the ansatz (8), $A_\mu = \hat{\Phi}^a A_\mu^a = 0$. Hence from Eq.(5), the Abelian electric field is zero and the Abelian magnetic field is independent of the gauge fields A_μ^a. To calculate for the Abelian magnetic field B_i, we rewrite the Higgs field of Eq.(8) from the spherical to the Cartesian coordinate system, [6]-[8]

$$\begin{aligned} \Phi^a &= \Phi_1 \, \hat{r}^a + \Phi_2 \, \hat{\theta}^a + \Phi_3 \, \hat{\phi}^a \\ &= \tilde{\Phi}_1 \, \delta^{a1} + \tilde{\Phi}_2 \, \delta^{a2} + \tilde{\Phi}_3 \, \delta^{a3} \end{aligned} \quad (17)$$

$$\text{where} \quad \begin{aligned} \tilde{\Phi}_1 &= \sin\theta \cos\phi \, \Phi_1 + \cos\theta \cos\phi \, \Phi_2 - \sin\phi \, \Phi_3 = |\Phi| \cos\alpha \sin\beta \\ \tilde{\Phi}_2 &= \sin\theta \sin\phi \, \Phi_1 + \cos\theta \sin\phi \, \Phi_2 + \cos\phi \, \Phi_3 = |\Phi| \cos\alpha \cos\beta \\ \tilde{\Phi}_3 &= \cos\theta \, \Phi_1 - \sin\theta \, \Phi_2 = |\Phi| \sin\alpha. \end{aligned} \quad (18)$$

The Higgs unit vector is then simplified to

$$\begin{aligned} \hat{\Phi}^a &= \cos\alpha \sin\beta \, \delta^{a1} + \cos\alpha \cos\beta \, \delta^{a2} + \sin\alpha \, \delta^{a3}, \quad (19) \\ \text{where,} \quad \sin\alpha &= \frac{\psi \cos\theta - R \sin\theta}{\sqrt{\psi^2 + R^2}}, \quad \beta = \frac{\pi}{2} - \phi, \end{aligned}$$

and the Abelian magnetic field is found to reduce to only the $\hat{r}_i$ and $\hat{\theta}_i$ components,

$$
\begin{aligned}
B_i &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right\}\hat{r}_i \\
&+ \frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial r} - \frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\phi}\right\}\hat{\theta}_i \\
&+ \frac{1}{r}\left\{\frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\theta} - \frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial r}\right\}\hat{\phi}_i, \\
&= -\frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\right\}\hat{r}_i + \frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial r}\right\}\hat{\theta}_i. \qquad (20)
\end{aligned}
$$

Defining the Abelian field magnetic flux as

$$
\Omega = 4\pi M = \oint d^2\sigma_i B_i = \int B_i(r^2\sin\theta d\theta)\hat{r}_i\, d\phi, \qquad (21)
$$

the magnetic charge enclosed by the sphere at infinity, M_∞, is calculated to be negative one,

$$
\begin{aligned}
M_\infty &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right)d\theta d\phi\bigg|_{r\to\infty} \\
&= -\frac{1}{2}\sin\alpha\bigg|_{0,r\to\infty}^{\pi} = -1. \qquad (22)
\end{aligned}
$$

From Eq. (22), we can conclude that the total magnetic charge M of these axially symmetric solutions does not depend on the degree of the Legendre polynomial m. Hence for all the solutions in the series the net magnetic charge of the whole system is always negative one. By letting M_0 to be the net magnetic charge when the radius of the enclosing sphere tends to zero at the origin, we can write,

$$
\begin{aligned}
M_0 &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right)d\theta d\phi\bigg|_{r\to 0} \\
&= -\frac{1}{2}\sin\alpha\bigg|_{0,r\to 0}^{\pi} = 1. \qquad (23)
\end{aligned}
$$

Similiarly, the net magnetic charge, M_0, at the point singularity of the solution is independent of the value of m. In fact, it is true that when $r < \sqrt[2m+1]{\frac{m+1}{m}}$, the topological magnetic charge is one, and when $r > \sqrt[2m+1]{\frac{m+1}{m}}$, the topological magnetic charge is negative one. Hence there is a 1-monopole located at $r = 0$.

We also notice that we can write the net magnetic flux per 4π sterad passing through the spherical surface of a partial enclosing sphere of radius r, sustaining an angle θ at the origin with the positive z-axis as

$$
\begin{aligned}
M_r(\theta) &= -\frac{(\psi(r)\cos\theta - R(\theta)\sin\theta)}{2\sqrt{\psi^2(r)+R^2(\theta)}}\bigg|_0^{\theta} \\
&= \frac{1}{2}\left\{\frac{-\psi(r)\cos\theta + R(\theta)\sin\theta}{\sqrt{\psi^2(r)+R^2(\theta)}} + \frac{\psi(r)}{|\psi(r)|}\right\}. \qquad (24)
\end{aligned}
$$

The A1 solution of Ref. [7] and [8], when $m = 1$, is the second member of this axially symmetric monopole solutions,

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\left\{\frac{2-r^3}{1+r^3}\right\}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}\tan\theta\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \frac{1}{r}\left\{\frac{2-r^3}{1+r^3}\right\}\hat{r}^a + \frac{1}{r}\tan\theta\,\hat{\theta}^a. \end{aligned} \tag{25}$$

It was first thought to consist of a 1-monopole at $r = 0$, surrounded by two antimonopoles located at the zeros of the Higgs field at $z = \pm\sqrt[3]{2} = \pm 1.2599$. However, a closer study of this solution reveals that the 1-monopole actually has a zero size MAM structure along the z-axis and hence possess a net unit magnetic charge. This MAM structure can be read from the plots of Eq.(24), $M_r(\theta)$ versus θ for the cases of $r \to \infty$, Fig.(1), and $r \to 0$, Fig.(2). Fig.(1) indicates that there is zero flux through the spherical shell at infinity when $\theta \neq \frac{\pi}{2}$ rad. Hence all the flux at infinity is radially inwards along the equatorial plane towards the origin giving a net topological charge of negative one for the $m = 1$ configuration. Fig.(2) shows that the net flux through the upper $(0 < \theta < \frac{\pi}{2})$ and lower $(\frac{\pi}{2} < \theta < \pi)$ spherical shell at small r is $+4\pi$ each and the flux through the circle when $\theta = \frac{\pi}{2}$ is -4π, hence indicating a MAM structure for the 1-monopole at the origin.

By plotting the magnetic field of this configuration we can confirm that at large r, all the magnetic field lies in the equatorial plane and is pointing radially inwards as the net magnetic charge M_∞ is -1. A plot of the magnetic field lines is shown in Fig.(3). The magnetic field lines do not converge at $r = 0$, showing the presence of a composite MAM 1-monopole. Hence the pole at the center of the composite monopole is an antimonopole surrounded by two structural monopoles at zero range from each other and yet they do not annihilate each. In fact this composite 1-monopole seems to have been "polarized" by the two outer antimonopoles at $z = \pm\sqrt[3]{2}$, leaving an antimonopole at its center. The antimonopoles situated at $z = \pm\sqrt[3]{2}$ form dipole pairs with the nearest monopoles of the MAM structure, thus screening off all the magnetic field above and below the equatorial plane at r infinity. There is no vortex ring in this configuration.

The vortex ring appears when $m = 2$, that is, when the gauge field potentials and Higgs field are respectively

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\left\{\frac{3-2r^5}{1+r^5}\right\}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}\left\{\frac{6\cos\theta\sin\theta}{3\cos^2\theta - 1}\right\}\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \frac{1}{r}\left\{\frac{3-2r^5}{1+r^5}\right\}\hat{r}^a + \frac{1}{r}\left\{\frac{6\cos\theta\sin\theta}{3\cos^2\theta - 1}\right\}\hat{\theta}^a. \end{aligned} \tag{26}$$

The plots of the magnetic flux, Eq.(24), versus θ, for values of r at infinity, Fig. (4), and $r \to 0$, Fig.(5), reveal that the composite 1-monopole at $r = 0$ is that of a MAMAM structure.

The two outer antimonopoles are located at the two zeros of the Higgs field at $z = \pm\sqrt[5]{3/2} = \pm 1.0845$, and the vortex ring is located at the ring of radius 1.0845 on the equatorial plane where the Higgs field vanishes, Fig.(6). The vortex ring is magnetically neutral and from Fig.(7), we can deduced that it consists of an inner negatively charged ring and an outer positively charged ring that are at zero distance from each other but yet do not annihilate the magnetic charge of each other. The magnetic field lines of this one vortex ring solution is shown in Fig.(8). The non-convergence of field lines at the origin, $r = 0$, indicates that the seemingly 1-monopole is not structureless. It seems from Fig.(8), that this composite 1-monopole has undergone "polarization" by the two outer antimonopoles at the $\pm$z-axis, as well as by the vortex ring.

The two vortex rings solution is the next solution of this series of axially symmetric monopole configurations with parameter $m = 3$. The gauge field potentials and Higgs field are respectively given by

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\left\{\frac{4-3r^7}{1+r^7}\right\}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{3\tan\theta}{r}\left\{\frac{5\cos^2\theta-1}{5\cos^2\theta-3}\right\}\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \frac{1}{r}\left\{\frac{4-3r^7}{1+r^7}\right\}\hat{r}^a + \frac{3\tan\theta}{r}\left\{\frac{5\cos^2\theta-1}{5\cos^2\theta-3}\right\}\hat{\theta}^a. \end{aligned} \tag{27}$$

As usual the two point antimonopoles are situated at the two zeros, $z = \pm\sqrt[7]{4/3} = \pm 1.0420$, of the Higgs field. The two vortex rings are located horizontally at $r = 1.0420$ and $\theta = 1.1071, \pi - 1.1071$ rad. Again from the plots of magnetic flux, $M_r(\theta)|_{r\to\infty}$, Fig.(9), and $M_r(\theta)|_{r\to 0}$, Fig.(10), of Eq.(24), together with the plot of the magnetic field lines, Fig.(11), we can conclude that the structure of the composite 1-monopole at the origin is MAMAMAM, with an antimonopole at the center. Hence by induction, we conclude that the number of A and M poles in the composite monopole is equal to $2m + 1$ and when m is even the pole in the center of the structure is a monopole and when m is odd we have an antimonopole in the center of the structure [6]. The number of vortex rings in the solution increases with m and is equal to $(m - 1)$.

4 COMMENTS

We have obtained exact axially symmetric A-M-A configurations of the SU(2) YMH theory which are characterized by a positive integer parameter m. The 1-monopole is located at the origin, $r = 0$ where the Higgs field is singular. The two outer antimonopoles are located at the two zeros of the Higgs field along the z-axis at $z = \pm\sqrt[2m+1]{(m+1)/m}$. When the parameter m exceeds unity, neutrally magnetic charge vortex rings start to appear around the z-axis. The number of vortex rings in the solution is equal to $(m - 1)$.

Further investigations reveal that the 1-monopole at the origin is not structureless but in fact corresponds to a zero size composite monopole lying along the z-axis. By induction we conclude that the number of poles in the composite monopole is given by $2m + 1$. When m is even, the center of the structure corresponds to a monopole and when m is odd, it corresponds to an antimonopole.

We have analysed the solutions when $m = 1, 2$, and 3, with zero size composite 1-monopole; MAM, MAMAM, and MAMAMAM, respectively.

Numerical static axially symmetric M-A-M-... chain at finite poles separations has also been discussed in Ref.[6]. These numerical solutions belong to the topologically trivial sector when the total number of poles and antipoles is even and to the topological unit sector when the total number of poles and antipoles is odd, said equal to $2m + 1$. We have only managed to find odd total number of poles and antipoles in our composite 1-monopole. Similar to the results of Ref.[6], we have a monopole at the center of the composite 1-monopole when m is even and an antimonopole in the center when m is odd. Also similar is that our solutions have zero magnetic dipole moment as the number of poles in our solutions is odd and the vortex rings are magnetically neutral.

Unlike the monopole solutions of Ref.[6], our monopoles and antimonopoles here are of unit charge only. We did not manage to get monopoles and antimonopoles of charge equal to two units for our axially symmetric monopoles solutions. In fact we have not found any M-monopoles with finite separations when $|M| \geq 1$.

We would also like to mention that for every monopoles, antimonopole, vortex rings solutions that we have discussed so far, there always exist an anti-configuration of the configurations discussed. This can be done by changing the ϕ winding number in the ansatz (8) from one to -1 and solving the Bogomol'nyi equation with the negative sign [8].

The structure of the composite monopole at the origin seems to come about as a result of "polarization" of the magnetic charge by the presence of the two outer antimonopoles along the z-axis and the vortex rings of the configuration, Fig.(3), Fig.(8), and Fig.(11).

We would also like to mention that the first member of this series of axially symmetric monopoles solution is the radially symmetric, Wu-Yang type of 1-monopole when $m = 0$, that is, $A^a_\mu = \frac{1}{r(1+r)}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right)$ and $\Phi^a = \frac{1}{r(1+r)}\hat{r}^a$. This Wu-Yang type monopole gauge potentials can be gauge transformed into the gauge potentials, $A'^a_\mu = \frac{-1}{1+r}\left(\hat{\rho}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) - \frac{1}{r}\tan\frac{\theta}{2}\ \delta^a_3\hat{\phi}_\mu$, and Higgs field $\Phi'^a = \frac{1}{r(1+r)}\ \delta^a_3$, by the gauge transformation $\omega = \exp(\frac{i}{2}\sigma_a\hat{\phi}^a\theta)$. Hence by using t' Hooft's definition, Eq.(5), the 1-monopole at $r = 0$, is a Dirac monopole with the Dirac string in the latter gauge.

We would also like to mention that one-half topological magnetic charge monopole is obtained when the parameter m is set to $-\frac{1}{2}$ in the solution, Eq.(14) [9].

5 Acknowledgements

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Figure Captions

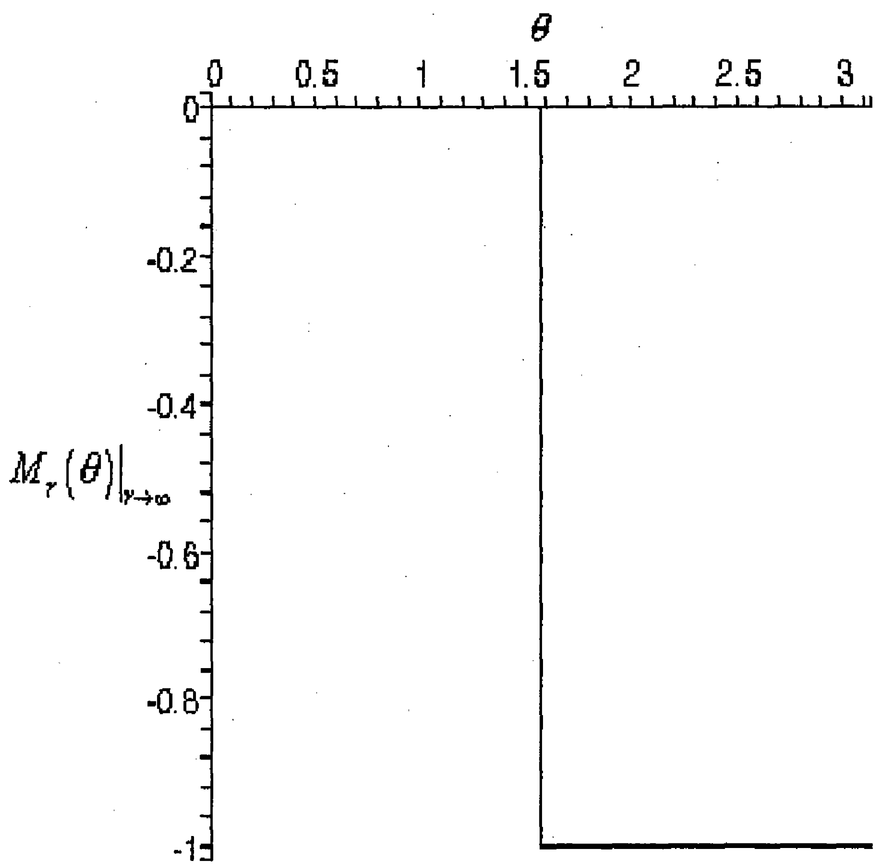

Figure 1: A plot of $M_r(\theta)$ at r infinity, when $m = 1$, versus θ. The total magnetic flux divided by 4π sterad covered by the partial sphere at various angles of θ clearly shows that the net magnetic charge of the system is -1.

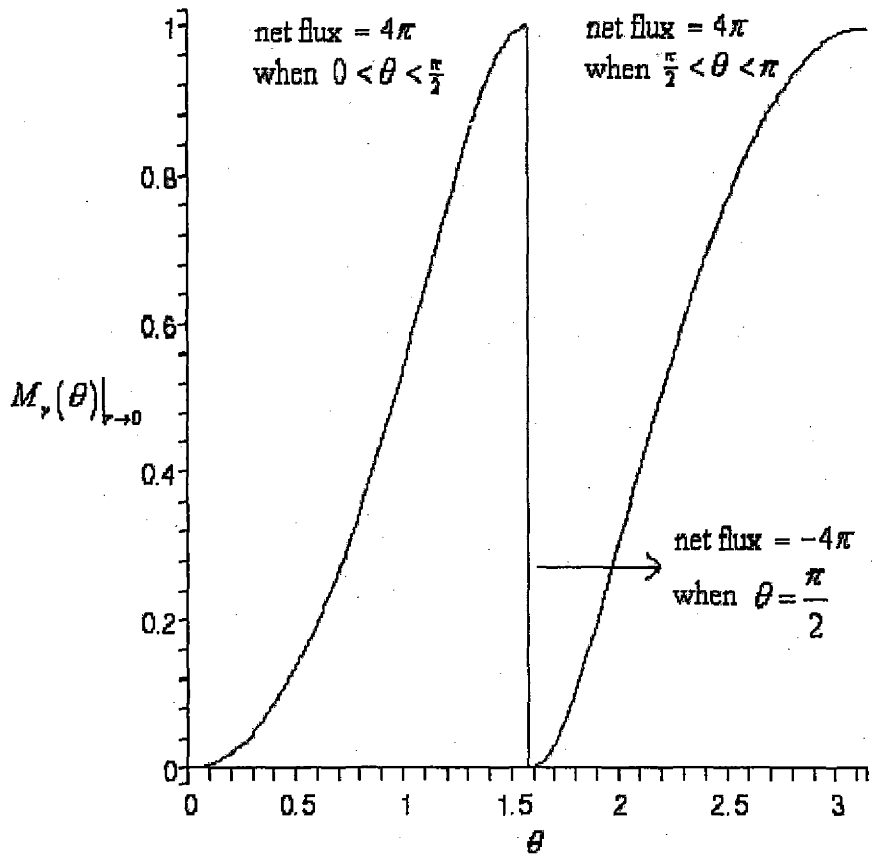

Figure 2: A plot of $M_r(\theta)$ at small $r < \sqrt[3]{2}$, when $m = 1$, versus θ. The total magnetic flux divided by 4π sterad covered by the partial sphere at various angles of θ clearly shows that the net magnetic charge of the system for small $r < \sqrt[3]{2}$ is positive one.

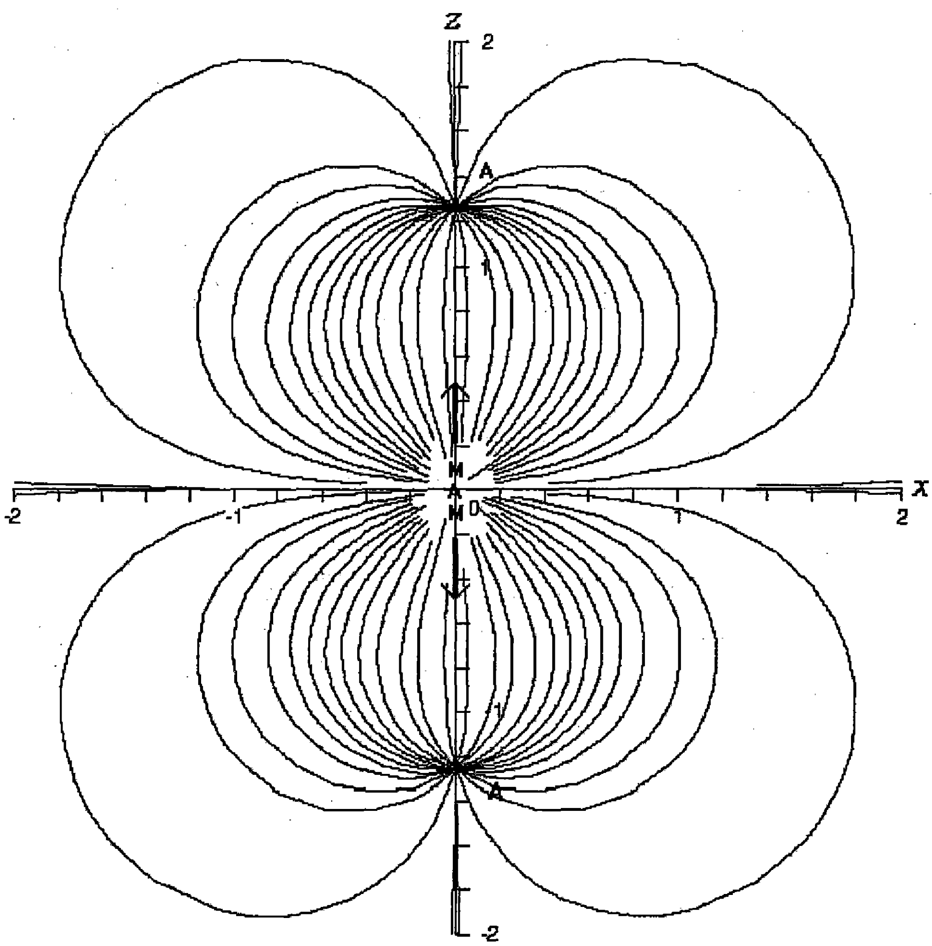

Figure 3: A plot of the magnetic field lines when $m = 1$ along a vertical plane through the z-axis. At large r, all the field lines are concentrated radially inwards in 2-D along the equatorial plane. The two antimonopoles that form a dipole pair with the nearest monopole of the zero size MAM structure are located along the z-axis at $z = \pm 1.2599$. The point structure at the origin is a composite MAM monopole of charge one.

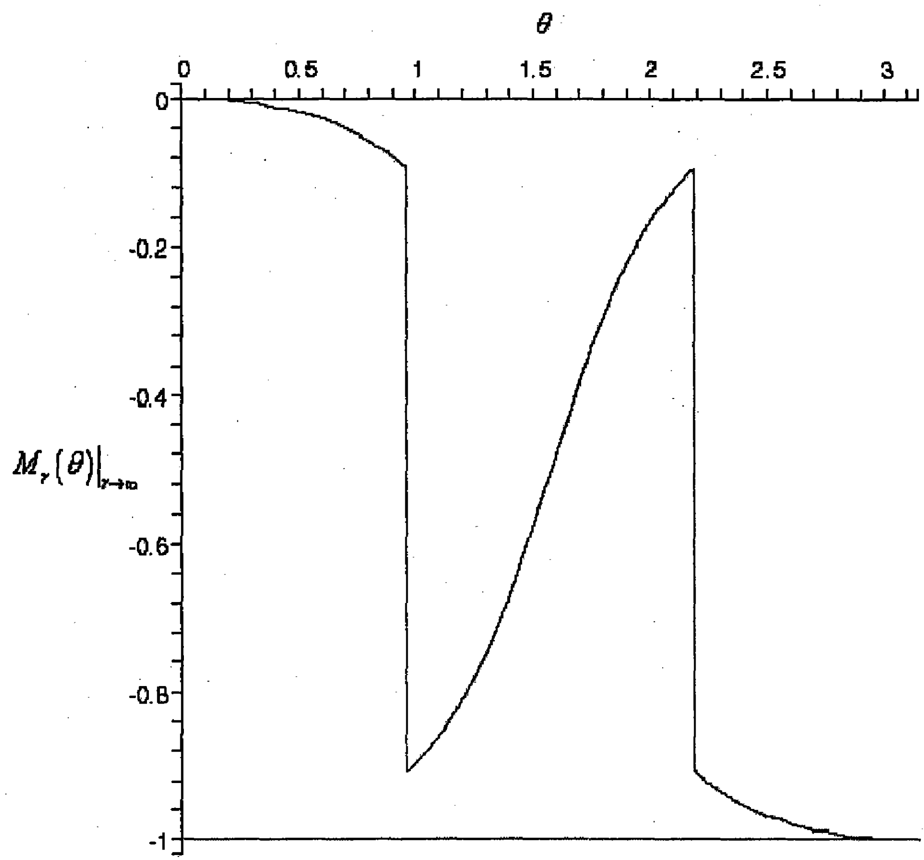

Figure 4: A plot of $M_r(\theta)$ at r infinity, when $m = 2$, versus θ. The total magnetic flux divided by 4π sterad covered by the partial sphere at various angles of θ clearly shows that the net magnetic charge of the system is -1.

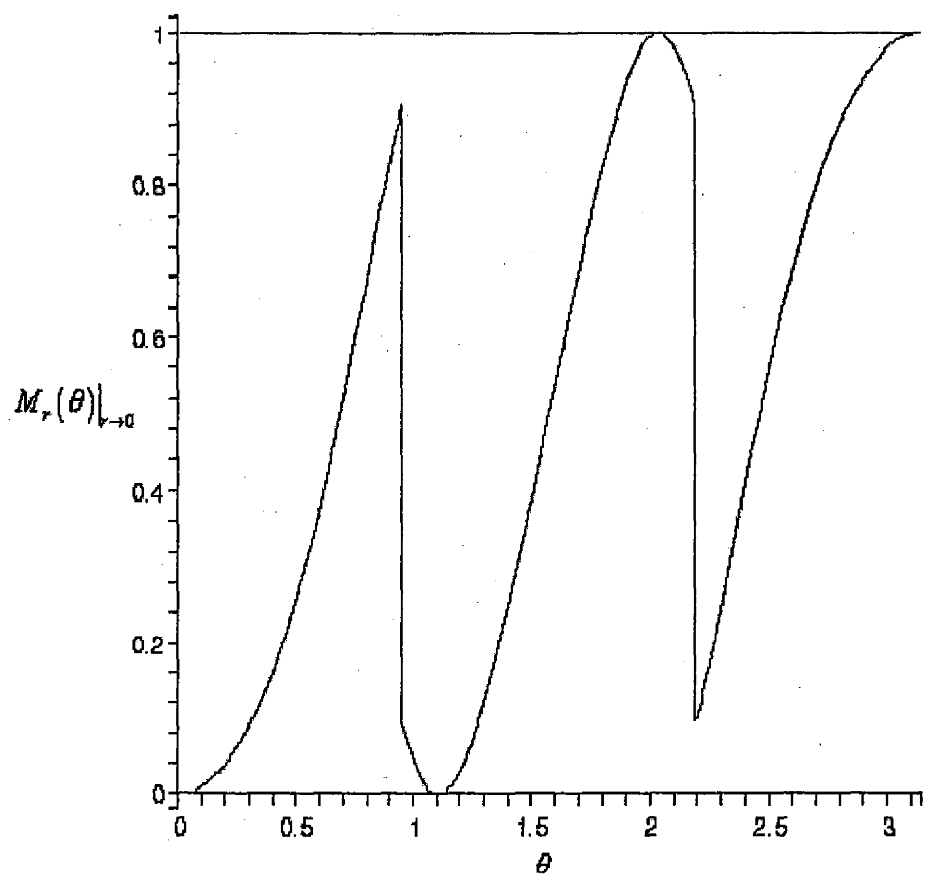

Figure 5: A plot of $M_r(\theta)$ at r close to zero, when $m = 2$, versus θ. The total magnetic flux divided by 4π sterad covered by the partial sphere at various angles of θ clearly shows that the net magnetic charge of the system for small r is positive one.

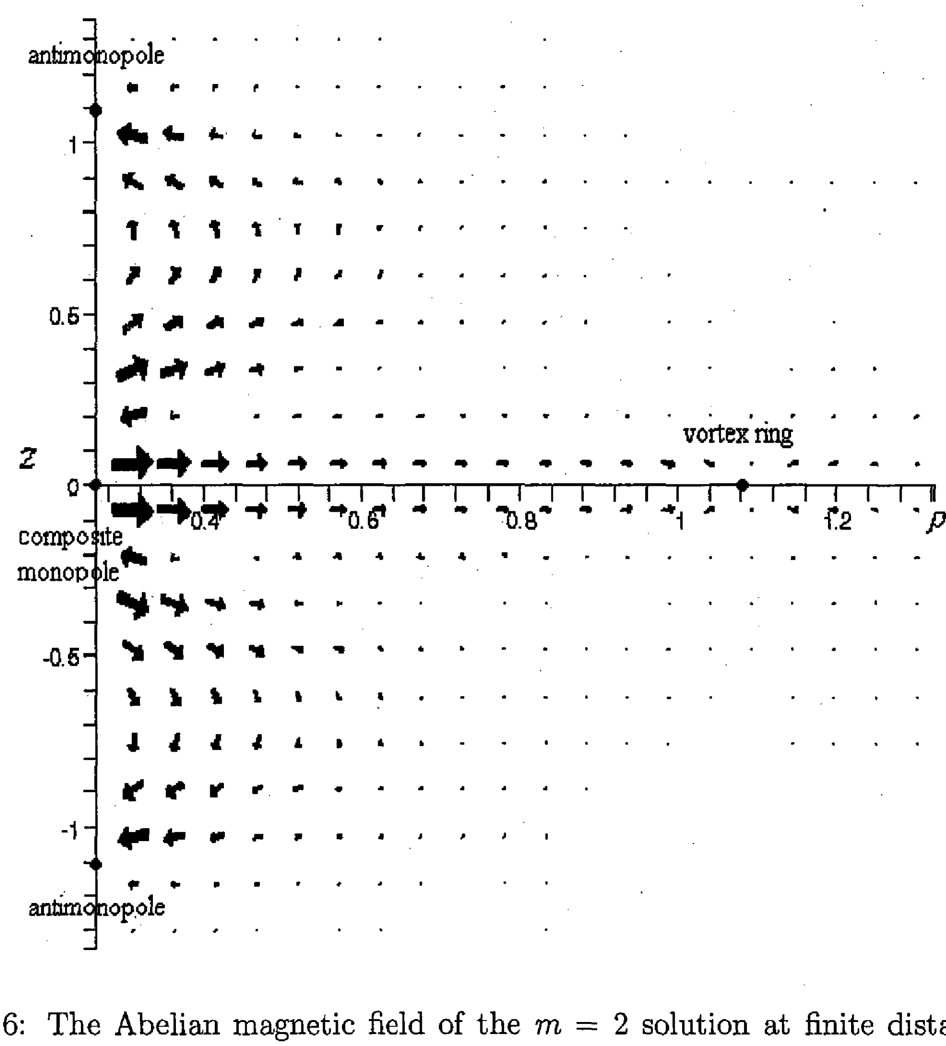

Figure 6: The Abelian magnetic field of the $m = 2$ solution at finite distances showing the presence of the two dipole pairs along the z-axis with opposite magnetic dipole moments and the vortex ring at $(z = 0,\ \rho = 1.0845)$.

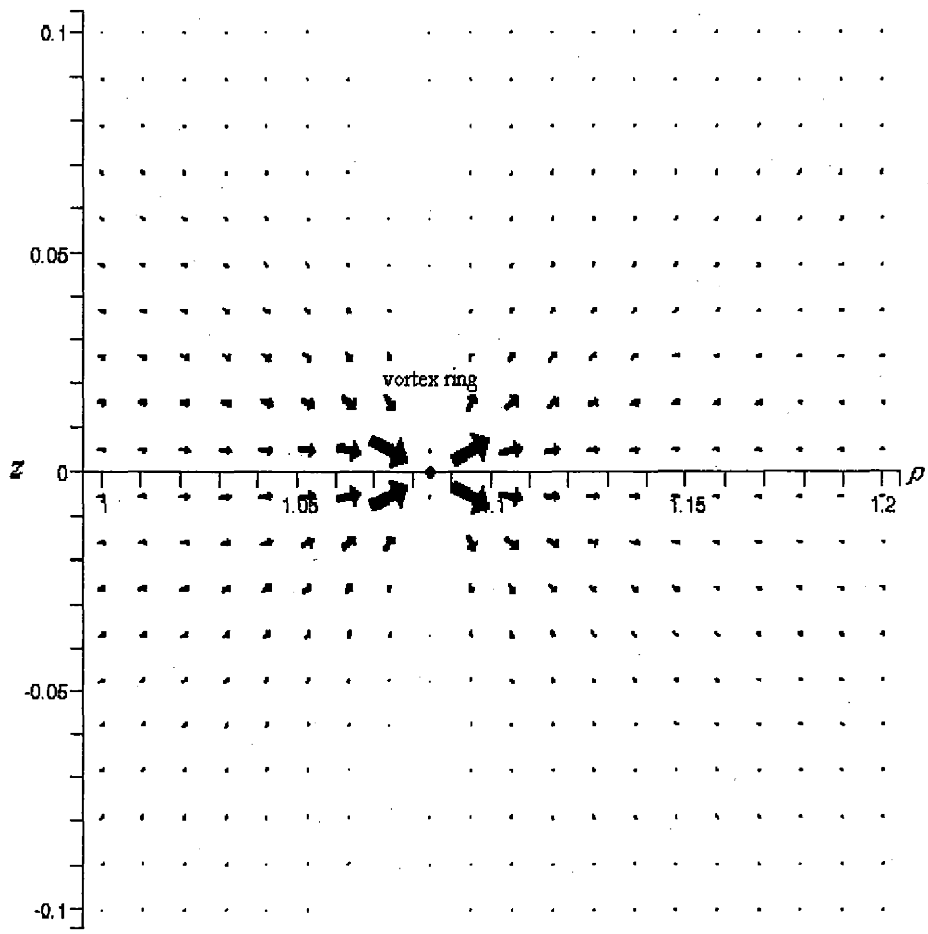

Figure 7: The Abelian magnetic field of the $m = 2$ solution at distances close to $(z = 0,\ \rho = 1.0845)$ showing the presence of a negatively charged inner vortex ring surrounded at zero distance by a positively charged outer vortex ring. The two in one oppositely charged vortex rings do not annihilate each other but form a single magnetically neutral vortex ring.

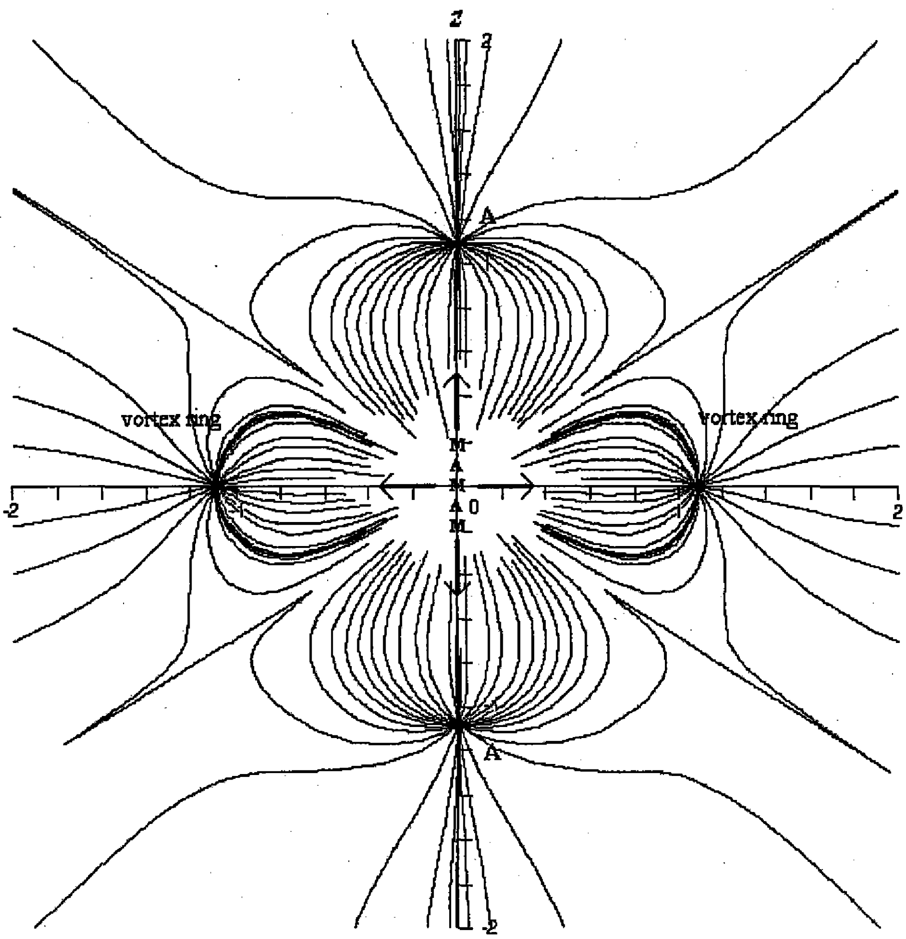

Figure 8: A plot of the magnetic field lines when $m = 2$ along a vertical plane through the z-axis. The vortex ring is situated horizontally at equal distances from the origin as the two antimonopoles at $z = \pm 1.0845$. The point structure at the origin is a composite MAMAM monopole of charge one.

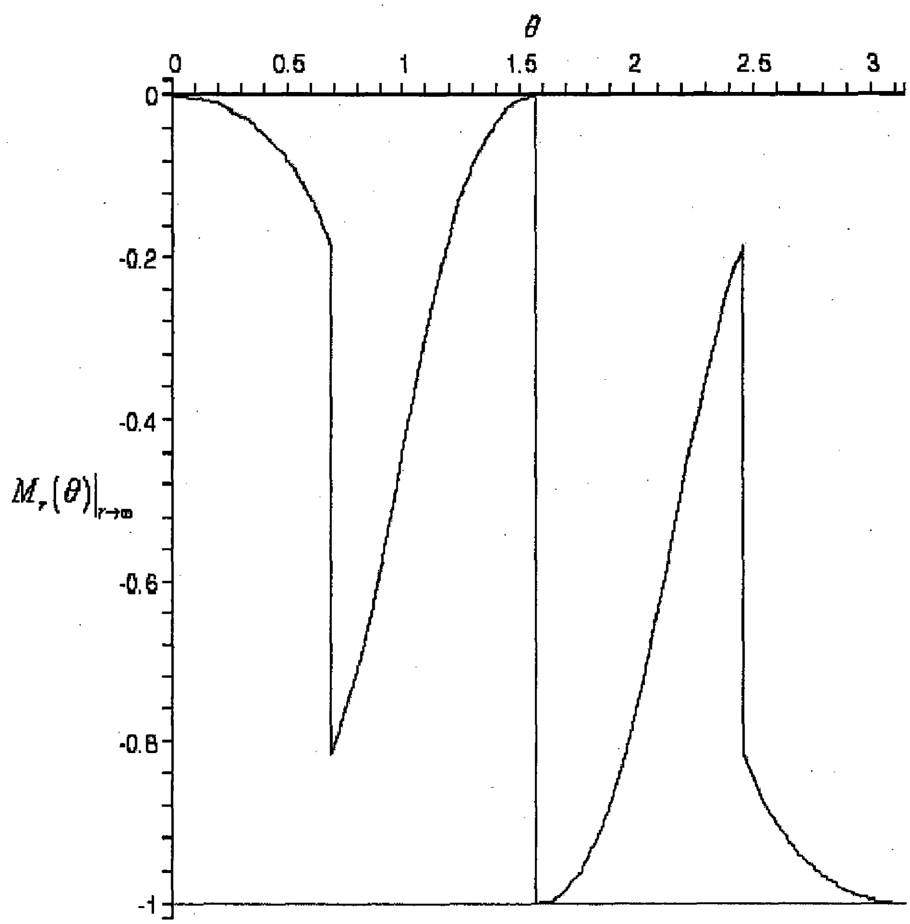

Figure 9: A plot of $M_r(\theta)$ at r infinity, when $m = 3$, versus θ. The total magnetic flux divided by 4π sterad covered by the partial sphere at various angles of θ clearly shows that the net magnetic charge of the system is -1.

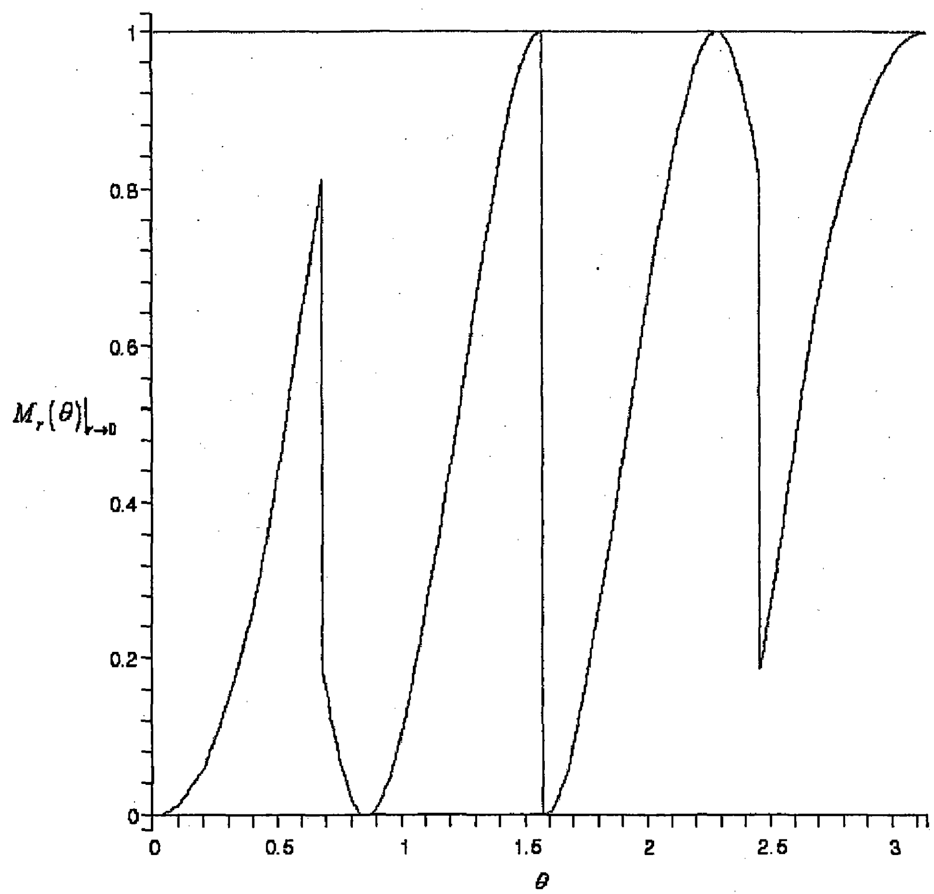

Figure 10: A plot of $M_r(\theta)$ at r close to zero, when $m = 3$, versus θ. The total magnetic flux divided by 4π sterad covered by the partial sphere at various angles of θ clearly shows that the net magnetic charge of the system for small r is positive one.

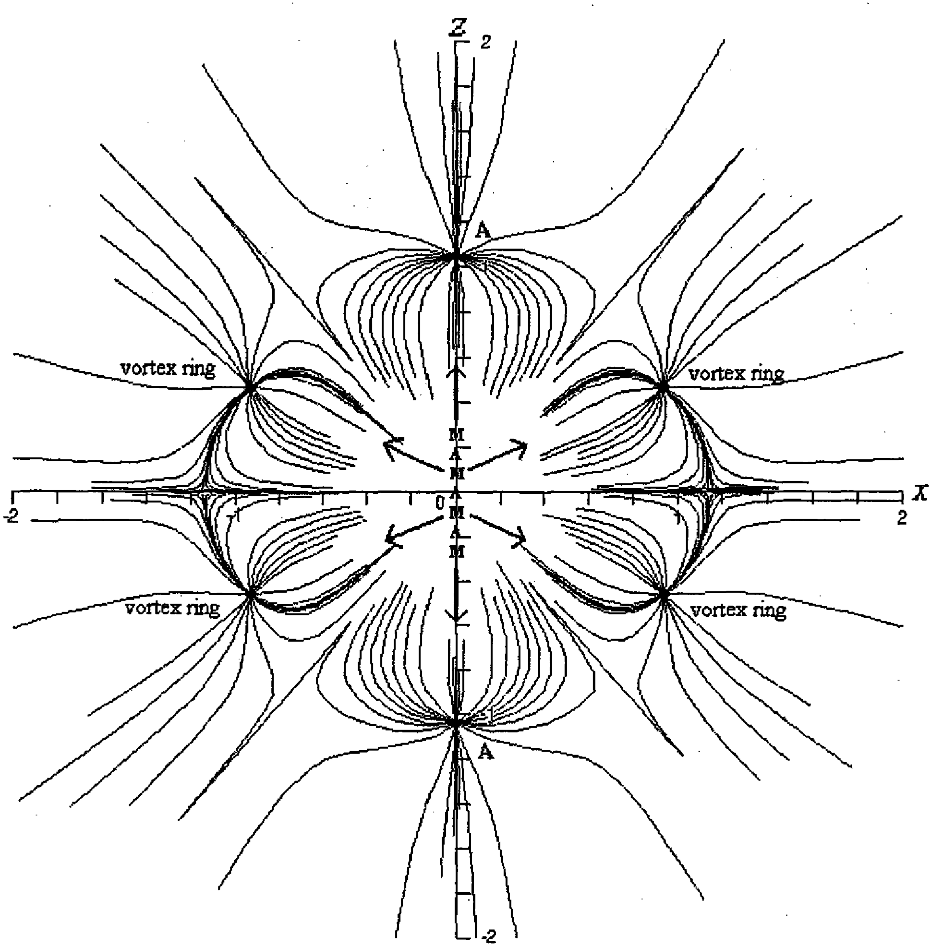

Figure 11: A plot of the magnetic field lines when $m = 3$ along a vertical plane through the z-axis. The two vortex rings are situated horizontally at equal distances from the origin as the two antimonopoles at $z = \pm 1.0420$. The point structure at the origin is a composite MAMAMAM monopole of charge one.