CLASSICAL MONOPOLES CONFIGURATION OF THE SU(2) YANG-MILLS-HIGGS FIELD THEORY

by

WONG KHAI MING

Thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

February, 2009

Acknowledgements

The author would like to make grateful acknowledgement to the following people:

1. My supervisor, Prof. Dr. Rosy Teh Chooi Gim for her continuous supervision, encouragement, guidance and advise.
2. The Dean and all staff members of the School of Physics for their assistance and advise.
3. The members of the theoretical research group of the School of Physics for their support.
4. The Academy of Sciences Malaysia for the award of the Scientific Advancement Grant Allocation (SAGA), account number: 304/PFIZIK/653004/A118.
5. The Ministry of Science, Technology and Innovation (MOSTI) for the award of the SCIENCEFUND, account number: 305/PFIZIK/613601.
6. My family for their support and encouragement.

Contents

List of Tables

List of Figures

List of Symbols

m – θ-winding number

n – ϕ-winding number

s – integer parameter in the axially and mirror symmetric exact solutions

κ – integer parameter in the boundary conditions for the numerical solutions

u – parameter in the axially and mirror symmetric exact solutions

μ – mass of the Higgs field

λ – strength of the Higgs potential

$T_{\mu\nu}$ – stress-energy tensor

$F^a_{\mu\nu}$ – gauge field strength tensor

A^a_μ – Yang-Mills gauge potential

Φ^a – Higgs scalar field

$\hat{\Phi}^a$ – unit vector of the Higgs scalar field

γ – arbitrary constant in the dyon solution

β – arbitrary constant in the dyon solution

P_u – Legendre polynomial of the first kind of degree u

Q_u – Legendre polynomial of the second kind of degree u

P_u^{u+s} – associated Legendre function of the 1st kind of degree u and order $u+s$

Q_u^{u+s} – associated Legendre function of the 2nd kind of degree u and order $u+s$

T^a – anti-Hermitian traceless matrices

L – Lagrangian

G – compact Lie group

σ^a – Pauli matrices

E – energy

$\hat{r}_i$ – spherical coordinate orthonormal unit vector

$\hat{\theta}_i$ – spherical coordinate orthonormal unit vector

$\hat{\phi}_i$ – spherical coordinate orthonormal unit vector

$\hat{r}^a$ – isospin coordinate orthonormal unit vector

$\hat{\theta}^a$ – isospin coordinate orthonormal unit vector

$\hat{\phi}^a$ – isospin coordinate orthonormal unit vector

$F_{\mu\nu}$ – Abelian electromagnetic field tensor

E_i – Abelian electric field

B_i – Abelian magnetic field

k_μ – topological magnetic current

Ω – Abelian magnetic flux

M – topological magnetic charge

M_0 – net magnetic charge enclosed by sphere at $r = 0$

M_∞ – net magnetic charge enclosed by sphere at r infinity

M_+ – net magnetic charge enclosed by upper hemisphere of infinite radius

M_- – net magnetic charge enclosed by lower hemisphere of infinite radius

M_A – net magnetic charge of the surrounding antimonopoles

Q – total electric charge of the dyon

Q_0 – net electric charge at the origin

ψ – profile function that depends on r

R – profile function that depends on θ

G – profile function that depends on θ and ϕ

K_1 – gauge field profile function that depends on r and θ

K_2 – gauge field profile function that depends on r and θ

K_3 – gauge field profile function that depends on r and θ

K_4 – gauge field profile function that depends on r and θ

ψ_1 – gauge field profile function that depends on r and θ

ψ_2 – gauge field profile function that depends on r and θ

R_1 – gauge field profile function that depends on r and θ

R_2 – gauge field profile function that depends on r and θ

Φ_1 – Higgs field profile function that depends on r and θ

Φ_2 – Higgs field profile function that depends on r and θ

X – compactified coordinate

a – numerical constant for the exact asymptotic solutions

b – arbitrary constant for the exact asymptotic solutions

KONFIGURASI EKAKUTUB KLASIK UNTUK TEORI MEDAN SU(2) YANG-MILLS-HIGGS

ABSTRAK

Teori medan SU(2) Yang-Mills-Higgs telah ditunjukkan bahawa ia mempunyai penyelesaian topologi penting yang mewakili ekakutub magnet dan multiekakutub. Walau bagaimanapun, penyelesaian tepat adalah terhad disebabkan penyelesaian ekakutub dan multiekakutub hanya boleh didapati di bawah limit potensi Higgs yang bernilai sifar. Ia juga telah ditunjukkan bahawa terdapat penyelesaian bukan-Bogomol'nyi yang tidak memenuhi persamaan Bogomol'nyi peringkat pertama tetapi hanya memenuhi persamaan medan peringkat kedua. Penyelesaian numerik ini boleh didapati di bawah limit potensi Higgs yang bernilai sifar dan juga potensi Higgs yang bernilai terhingga, dan mereka mewakili sistem rangkaian ekakutub-antiekakutub dan gelang vorteks.

Dalam tesis ini kami menyelidik teori medan SU(2) Yang-Mills-Higgs untuk memperolehi lebih banyak penyelesaian ekakutub klasik dan berharap pengetahuan dan pengalaman yang diperolehi di peringkat klasik akan membolehkan kita memahami dengan lebih mendalam seluruh struktur teori medan 'gauge' dan sifat-sifat ekakutub magnet. Dengan menggunakan ansatz yang spesifik dan di bawah limit potensi Higgs yang bernilai sifar, kami memperolehi penyelesaian tepat dan juga mengkaji penyelesaian numerik. Penyelesaian kami boleh dibahagikan kepada penyelesaian dengan simetri paksi dan simetri cermin sepanjang paksi z. Kesemua penyelesaian tepat adalah memenuhi persamaan Bogomol'nyi

peringkat pertama tetapi mempunyai tenaga tak-terhingga. Oleh itu mereka merupakan penyelesaian Bogomol'nyi-Prasad-Sommerfield (BPS) yang berlainan.

Penyelesaian dengan simetri paksi adalah terdiri daripada (i) Ekakutub-antiekakutub dan gelang vorteks, (ii) setengah ekakutub, (iii) Ekakutub penuh dan setengah ekakutub, (iv) Dion bercas setengah ekakutub, (v) Pasangan ekakutub-antiekakutub (MAP) dan (vi) Ekakutub BPS tepat. Sebaliknya, penyelesaian dengan simetri cermin adalah merangkumi (vii) Penyelesaian multiekakutub siri C, (viii) Siri multiekakutub 2s, (ix) Penyelesaian skrin 1s. Di sini juga wujud anti-konfigurasi untuk semua penyelesaian tepat di mana arah medan magnet Abelian dan tanda cas kutub telah diterbalikkan.

CLASSICAL MONOPOLES CONFIGURATION OF THE SU(2) YANG-MILLS-HIGGS FIELD THEORY

ABSTRACT

The SU(2) Yang-Mills-Higgs field theory has been shown to possess important topological solutions which represents magnetic monopoles and multi-monopole. However, exact solutions are limited as the monopole and multi-monopole solutions are only exactly solvable under the limit of vanishing Higgs potential. It has also been proved the existence of non-Bogomol'nyi solutions that do not satisfy the first order Bogomol'nyi equations but only the second order field equations. These numerical solutions exist both in the limit of vanishing Higgs potential as well as in the presence of a finite Higgs potential, and they represents systems of monopole-antimonopole chains and vortex rings.

In this thesis we study the SU(2) Yang-Mills-Higgs field theory for more classical monopoles solutions, with the hope that insights and experiences gained at the classical level will illuminate our understanding of the whole structure of the gauge field theories, as well as the properties of magnetic monopoles. By using a modified ansatz and within the limit of vanishing Higgs potential, we obtained exact solutions and also studied numerical solutions. Our solutions can be divided into solutions with axial symmetries and mirror symmetries along the z-axis. All the exact solutions satisfy the first order Bogomol'nyi equations but possess infinite energies. Hence they are a different kind of Bogomol'nyi-Prasad-

Sommerfield (BPS) solutions.

The axially symmetric solutions consists of (i) antimonopole-monopole-antimonopole (A-M-A) and vortex rings, (ii) one-half monopole, (iii) a full and one-half monopole, (iv) dyons of one-half monopole charge, (v) the numerical monopole-antimonopole pair (MAP) solutions and (vi) the exact BPS one monopoles. On the other hand, the solutions with mirror symmetries includes (vii) C series multimonopole solutions, (viii) 2s multimonopole series and (ix) 1s screening solutions. There also exist an anticonfiguration to all the exact solutions where the directions of its Abelian magnetic field and hence its topological magnetic charge sign are reversed.

Chapter 1

Introduction

1.1 Particle Physics and Gauge Theory

Particle physics is a branch of physics that studies elementary constituents of matter and the interactions between them. These include atomic constituents such as electrons, protons, neutrons, quarks (constituent of protons and neutrons), particles produced by scattering and radiative processes (photons, neutrinos and muons), as well as some exotic particles. The fundamental interactions between all the above particles are well-known to us: the electromagnetic, weak, strong and gravitational interactions. Particle physics is sometimes called 'high energy physics', because some of the elementary particles do not occur under normal circumstances in nature, and can only be experimentally created and detected during high energy collisions in particle accelerators. Nowadays theoretical studies of high energy physics are carried out with non-Abelian gauge theories. Hence we will start with discussing what is gauge theories all about and exploring its brief historical development. Before we move on, we would like to emphasize that readers are assumed to have a basic understanding in particle physics.

Gauge theories are a class of physical theories based on the idea that symmetry transformations can be performed locally as well as globally, with the main object of study is the gauge field (for a detail primer on gauge theory, readers are referred to Moriyasu (1983)). The earliest idea of gauge theory was found in Maxwell's electrodynamics. Hermann Weyl (1919) attempted to describe the electromagnetic interaction by using the analogy of connection in general relativity. He conjectured that invariance under the change of scale ('gauge') might also be a local symmetry of the theory of general relativity. Unfortunately, this conjecture was later pointed out that it would lead to conflict with known physical facts.

However, development of the quantum mechanics revived Weyl's gauge theory of electromagnetism. With some modifications by replacing the scale factor with a complex quantity, and turning the scale transformation into a change of phase (a U(1) gauge symmetry), it was realized by Weyl (1929), Fock (1927) and London (1927) that this could provide a neat explanation for the effect of an electromagnetic field on the wave function of a charged quantum mechanical particle. It was then clear that electromagnetic interaction of charged particle could be intepreted as a local gauge theory within the framework of quantum mechanics, in the language of quantum electrodynamics (QED). The gauge transformation is actually the transformation of the phase of the wavefunction, which depends on the space-time location. The gauge group is the group of all possible gauge transformation and the gauge group associated with electromagnetism is the U(1) group.

The study of gauge theory on Abelian electrodynamics is considered the old period of gauge theory. The new period of gauge theory begins in 1954 with the pioneering effort of Yang and Mills (1954) to extend the gauge symmetry beyond the narrow limits of electromagnetism. Yang and Mills (1954) introduced non-Abelian gauge theories as models to understand the strong interaction holding together nucleons in atomic nuclei. Generalizing the gauge invariance of electromagnetism, they attempted to construct a theory based on the action of the non-Abelian SU(2) symmetry group on the isospin doublet of protons and neutrons, similar to the action of the U(1) group on the spinor fields of quantum electrodynamics. Although the original purpose of Yang and Mills was not fulfilled, Yang and Mills' effort established the foundation for modern gauge theory and stimulated world wide research effort on gauge theories since then.

The idea of non-Abelian gauge symmetry later found application in the quantum field theory of the weak force, and its unification with electromagnetism in the electroweak theory. Glashow (1961) constructed an SU(2) $\times$ U(1) model along these lines which had many attractive features, but is without the vital symmetry breaking Higgs fields. Weinberg (1967) and Salam (1968) then introduced the Higgs field into the SU(2) $\times$ U(1) model and the resulting field theory has turned out to be extraordinarily successful. This successful theory is known as Weinberg-Salam model and has convinced most physicist that non-Abelian gauge theories of the weak and electromagnetic interactions are good physical theories. One of the most important predictions of the electroweak force, namely the existence of three heavy gauge bosons Z_0, W_+ and W_- with energies of the order 100 GeV, was confirmed by their discovery in accelerator experiments at CERN in 1983.

After the success of the Weinberg-Salam model to desribe two fundamental intereactions (weak and eletromagnetic) in one language, the next is the strong interaction, a fundamental force describing the interactions of the quarks and gluons found in hadrons (such as the proton, neutron or pion). The relevant theory for strong interactions emerged is now known as Quantum Chromodynamics (QCD). It is a gauge theory with the action of the SU(3) group on the color triplet of quarks. Hence it is also sometimes called 'color gauge theory'. The gauge symmetry are unbroken, so there are no Higgs field present. A huge body of experimental evidence for QCD has been gathered over the years and proves the consistency of QCD as the language for strong interaction. QCD enjoys two peculiar properties: a) asymptotic freedom, and b) confinement.

Asymptotic freedom means that in reactions at very high energy scale, quarks and gluons interact very weakly. This behavior was first discovered in the early 1970s by Gross and Wilczek (1973) and also by Politzer (1973). For this work they were awarded the 2004 Nobel Prize in Physics. Confinement, on the other hand, means that the force between quarks does not diminish as they are separated. In fact, they get even stronger. Because of this, it would take an infinite amount of energy to separate two quarks. There are assumptions that it is the color charge of quarks that is being confined. Following this behavior, the quarks are forever bound into hadrons and cannot exist outside a hadron as free particles. Although analytically unproven, confinement is widely believed to be true because it explains the consistent failure of free quark searches, and it is easy to demonstrate in lattice QCD.

The combination of Weinberg-Salam model and QCD is generally known as the Standard Model. It describes three of the four fundamental forces in our universe, the strong, weak, and electromagnetic fundamental forces, by using mediating gauge bosons. The species of gauge bosons are the gluons, W- and W+ and Z bosons, and the photons, respectively. The Standard Model then has 40 species of elementary particles (24 fermions, 12 vector bosons, and 4 scalar bosons), which can combine to form composite particles, accounting for the hundreds of other species of particles discovered since the 1960s. Finally, the existence of the gauge boson known as the Higgs boson, is yet to be conclusively confirmed, but will be probed in experiments at a higher energy scale.

Hence, it is obvious that the power of gauge theory stems from the extraordinary success of the mathematical formalism in providing a unified framework to describe the quantum field theories of electromagnetism, the weak force and the strong force, and possibly include gravity to describe all the fundamental interactions in one language. Although a quantum theory for gravity has yet to be successfully set up but there are strong believes that it will emerge in the future.

Even some formulations of general relativity are a form of gauge theories, in one way or another. The Standard Model has also been found to agree with almost all the experimental tests conducted to date. However, most particle physicists believe that it is an incomplete description of Nature, and that a more fundamental theory awaits discovery. This will be addressed in the next section.

1.2 Beyond the Standard Model

As stated in the previous section, although the Standard Model are extremely successful by providing a very good description of phenomena observed by experiments, it is an incomplete theory. The reason is that there are phenomena that are not accurately described by this theory. For example, even though physicists knew the masses of all the quarks except for top quark for many years, they were simply unable to accurately predict the top quark's mass without experimental evidence because the Standard Model lacks an explanation for a possible pattern for particle masses. The Standard Model is also as yet unable to explain gravity in terms of particles. Furthermore, a series of open questions demand for a more complete theory. Are quarks and leptons actually fundamental, or made up of even more fundamental particles? Why are there exactly three generations of quarks and leptons? Why do we observe matter and almost no antimatter if we believe there is a symmetry between the two in the universe? Why can't the Standard Model predict a particle's mass?

However, this does not mean that Standard Model is wrong, but one needs to go beyond the Standard Model in the same way that Einstein's Theory of Relativity extended Newton's laws of mechanics. One needs to extend beyond the Standard Model with something totally new in order to thoroughly explain mass, gravity and other phenomena. This area of research is often described by the term ‘Beyond the Standard Model’ and it studies possible extensions to the Standard Model that will be probed in up-coming experiments. There are many problems where beyond the Standard Model tries to tackle, such as the hierarchy problem, dark matter, the cosmological constant problem, the strong charge parity (CP) problem. It is not possible to give a complete description on these issues here, but we will briefly discuss two of the most actively studied areas, which are supersymmetry and string theory (this phrase is often used as shorthand for superstring theory, as well as related theories such as M-theory).

Supersymmetry (often abbreviated SUSY) was originally proposed by Wess and Zumino (1973). It is a symmetry that relates elementary particles of one spin to another particle that differs by half a unit of spin and these related particles are known as superpartners. In other words, every fundamental fermion has a superpartner which is a boson and vice versa. For example, for every type of quark there may be a type of particle called a ‘squark’. Since the particles of the Standard Model do not have this property, supersymmetry must be a broken symmetry allowing the ‘sparticles’ to be heavy. Readers are referred to the textbook by Ferrara (1987) for a complete account on supersymmetry.

The first realistic supersymmetric version of the Standard Model was proposed by Dimopoulos and Georgi (1981) and is called the minimal supersymmetric Standard Model (MSSM). It was introduced in order to solve the Hierarchy Problem, that is, to explain why particles not protected by any symmetry (like the Higgs boson) do not receive radiative corrections to its mass driving it to the larger scales (GUT, Planck...). No supersymmetric particle has yet been found, but supersymmetry is expected to be observed by experiments at the Large Hadron Collider. Supersymmetry also seems to have other interesting properties: its gauged version is an extension of general relativity (supergravity), and it is a key ingredient for the consistency of most versions of string theory. Another advantage of supersymmetry is that supersymmetric quantum field theory can sometimes be solved.

It is said earlier that in the daunting task to unify all the known forces in universe, it is still an opened question on how to implement gravity in the scheme of quantum field theory. Similar to the other interactions, gravitational interaction should be mediated by a spin-2 gauge boson, more oftenly called graviton. A lot of experimental detection has been put up to probe the gravitons but there are no conclusive results yet. Even the quantum field theory of gravity runs into trouble. Investigation of the gravitational interaction at short distances revealed that the mathematical description gives unavoidable divergencies. The solutions to this serious problem seems to be the popular string theory.

String theory is a fundamental physics model with argument based on the fact that particles are no longer zero-dimensional object but rather one-dimensional extended strings with the length of the order of the Planck length 10^{-33} cm. There, the particles are identified with the vibrational modes of these strings. By replacing the point-like particles with strings, an apparently consistent quantum theory of gravity emerges. Moreover, it might be possible to 'unify' all the known fundamental forces by describing them with the same set of equations. This shows that the superstring theory is (at the moment) the most promising candidate for a Theory of Everything (TOE). However, superstring needs to be verified experimentally to be scientifically valid. Not many experimental evidence has been obtained. Nevertheless, with the construction of the Large Hadron Collider in CERN, scientists remains upbeat with the hope to produce relevant data to support superstring theory, though it is believed that any theory of quantum gravity would require much higher energies to probe directly.

There are originally five different consistent superstring theories. In the early 1990s, it was shown that the various superstring theories were related by dualities. This allows one to relate the description of an object in one superstring theory to the description of a different object in another superstring theory. Inspired by these relationships, during a conference at University of Southern California in 1995, Witten proposed that each of the super string theories is a different aspect of a single underlying theory, the so-called 'M-theory'. Studies of string theory have revealed that it predicts higher-dimensional objects called branes and also suggests the existence of ten or eleven (in M-theory) spacetime dimensions, as opposed to the usual four (three spatial and one temporal) used in relativity theory. However, the theory can describe universes with four effective (observable) spacetime dimensions by a variety of methods. An important branch of the field deals with a conjectured duality between string theory as a theory of gravity and gauge theory. It is hoped that research in this direction will lead to new insights on quantum chromodynamics.

1.3 Classical Gauge Theory

Study of gauge field theories can actually be divided into classical and quantum parts. Phenomenological models of the Yang-Mills theory described in the above section belongs to the quantum Yang-Mills theory. Despite the tremendous success in the quantum aspect of Yang-Mills theory, classical Yang-Mills theories play a role which is of no less importance than quantum Yang-Mills theories. If everything about the classical field configurations is understood, then in principal all question concerning the corresponding quantum field theories can be answered (Actor, 1979). Even if partial information is retrieved, it would be useful for the construction of a complete quantum field theory. This is the basic hope which motivates research activities in classical Yang-Mills theories, as classical and quantum theory progress in parallel, with the classical information acting as a supporting platform for the quantum part.

Studying solutions to the classical gauge field theories is then an interesting pursuit. There are actually a huge number of classical solutions to the non-Abelian Yang-Mills gauge theory. However, due to the natural limitations of this thesis, we are not able to give a full account for all these solutions. We will only concentrate on briefly discussing some of the more important classical Yang-Mills solutions. These include the Euclidean space solutions (instanton and meron) and the Minkowski space solutions (monopole and dyon). Monopole is of course the more interesting one to us and will be discussed in detail in the next chapter.

However, before we go any further, we would like to briefly discuss some development of classical gauge theory. For a long period starting from 1956, there were not much progress made in the classical gauge theory. The first exact solution of the classical pure SU(2) YM theory was found by Ikeda and Miyachi (1962). However this solution attracts little interest as it is only imbedded Coulomb solution into the SU(2) theory. A more genuine non-Abelian Yang-Mills solution was later discovered by Wu and Yang (1968). A very interesting feautre of the Wu-Yang solution is that it describes a point-like non-Abelian magnetic monopole and does not possess a string. Hence it seems that Yang-Mills theory provides a natural stage for one to look for magnetic monopole solution.

Important progress was made by Nielson and Oleson (1973), who introduced a 'classical Higgs mechansim' into classical gauge theory. This mechanism is quite analogous to quantum field theoretic Higgs mechanism. It causes the classical gauge field to become 'massive' in the sense that certain components of the gauge potential must be short range (decreasing as $\exp(-Mr)$ at large r) for the energy to be finite. In their example, Nielson and Oleson used two Higgs field triplets to make all gauge field components decrease exponentially away from an axis. Hence the gauge field is essentially contained within a tube or 'vortex'. Nielson and Oleson's work directly inspired another important discovery, which is the very important finite energy 't Hooft-Polyakov magnetic monopole solution.

We will discuss this in the next chapter.

We would also like to point out that classical Yang-Mills theory can actually be studied independently of exact solutions (Actor, 1979). Classical results gained will improve the path-integral formulation of the quantum Yang-Mills theory. However we do not follow this path in this thesis but concentrate in exploring the properties of the important classical solutions. We will discuss in detail some of the important solitonic solutions of the classical Yang-Mills gauge theory in the next chapter before exploring our version of solutions.

1.4 Euclidean Space Solutions

Before we turn to discuss the monopole solution, we would like to give a brief account on Euclidean space solutions here. Some of the well-known Euclidean space solutions are the instanton and meron solutions. The instanton is a self-dual solution of the Yang-Mills equations. It is interpreted as a tunnelling process between two vacuum states of a quantum mechanical system (Callan *et al.*, 1976; Jackiw and Rebbi, 1976a, 1976b; 't Hooft, 1976b). Belavin *et al.* (1975) found the one instanton solution whereas the multi-instanton solutions were obtained by 't Hooft (1976a), Jackiw *et al.* (1977) and Witten (1977) separately.

The N-instanton solution is a vacuum fluctuation with N units of topological charge. The main properties of the instanton are that it is nonsingular and localized in all direction in E_4 including the imaginary time axis. Since it is a self-dual solution, it has vanishing energy and momentum density. This implies that the instantons are not particles but corresponds to a vacuum tunnelling event in the Minkowski space. This vacuum tunnelling event in turn implies that there are more than one vacuum state in the theory. In fact there are denumerable infinity of vacuum as there exist instanton solution with arbitrary large topological charge q. By labelling a vacuum state with topological index n by $|n>$, then the N-instanton solution will correspond to a tunnelling from the vacuum state $|n>$ to the vacuum state $|n+N>$.

Another type of Euclidean Yang-Mills solution are the meron solutions. A meron is a pointlike concentration of one half unit of topological charge. Unlike instanton, meron solutions are not self-dual and because of their singular nature meron solutions have infinite action, which makes their physical relevance somewhat obscure. However, multimeron has been shown to exist (the only known explicit solution describe two merons or a meron and an antimeron). No explicit expression for the multi-meron solutions have been found and the only known exact meron solutions are the one and two-meron solutions of de Alfaro *et al.* (1976).

Similar to instantons, merons also correspond to vacuum tunnelling events in the Minkowski space. However an instanton tunnels between two Gribov vacua with topological index $n = \pm\frac{1}{2}$ in the Coulomb gauge whereas a meron tunnels between the vacuum state with topological index $n = 0$ and a Gribov vacuum with $n = \frac{1}{2}$. The Gribov vacua were discovered by Gribov (1977) when he noticed that the gauge potential A^a_μ in the non-Abelian gauge theory is not uniquely determined in the Coloumb gauge. As a result of this ambiguity, the SU(2) Yang-Mills theory has three rotationally symmetric vacua with topological index $n = 0$ and $n = \pm\frac{1}{2}$ (Gribov vacua). The degeneracy of these Coulomb gauge vacua is removed by the merons and not the instantons.

The vacuum tunnelling process of the merons and instantons seem to support the conjecture of Callan *et al.* (1977, 1978b) that an instanton consist of two merons. For small coupling the merons are bounded together in pairs to form instantons. However when the coupling becomes large enough, the instatons will start to dissociate into merons and the theory goes into the confining phase. Since quark confinement problem porposed by Callan *et al.* (1977, 1978a) is based on semi-classical arguments, the overall Yang-Mills coupling must be fairly weak otherwise the whole arguments break down.

Chapter 2

Monopole, Multimonopole and Dyon

2.1 Magnetic Monopole

Apart of the Euclidean solutions explained in previous chapter, we are more interested in the Minkowski space solutions, which is the magnetic monopole (and also dyon). Because of its importance, we devote some pages for detail explanations on the magnetic monopole, including a brief historical development on the studies of monopole. Analogous to electric charge, a magnetic monopole is a particle that may be generally described as 'a magnet with only one pole'. While the Maxwell equations in vacuum are symmetric under a duality transformation between the electric and magnetic field, this is no longer true in the general case (when higher symmetry is considered). The simple reason is the absence of the magnetic monopole in the theory. However, there are optimists who believe that a physical theory should possess a symmetry between electricity and magnetism.

Historically, the first effort to study monopole had been made by Dirac (1931) where he constructed an Abelian point-like monopole which contained a singular string. One end of the string extends to infinity while a magnetic monopole is situated at the other end. The quantized version of Dirac's theory can solve an up-until-then an open problem in physics: the quantization of the electric charge. The result shows that the product between the single electric and the single magnetic charge is proportional to an integer. This suggests that, if there exist a single magnetic monopole in the universe, then electric charge is quantized, and vice versa.

As stated in chapter one, Nielson and Oleson (1973) made important progress by introducing a 'classical Higgs mechanism' into classical gauge theory. Following this line, 't Hooft (1974) and Polyakov (1974, 1975) made important breakthrough in the monopole theory by constructing classical solutions possessing the properties of magnetic monopoles, in the framework of Georgi-Glashow model. Preskill (1984) has emphasized that the essence of this breaktrough is that while a Dirac monopole could be incorporated in an Abelian theory, some non-Abelian models (i.e. the Georgi and Glashow model) inevitably contain monopole-like solutions. Different to Dirac's monopole, the 't Hooft-Polyakov solutions represent extended, localized and finite energy magnetic monopoles with topological stability. This spherically symmetric monopole solution is one of the most important solutions obtained so far.

Moreover, particle field theories such as Grand Unified Theories (GUTs) also predict the existence of monopole solutions on mathematical ground. Whenever there is an unbroken U(1) symmetry the existence of magnetic monopoles is unavoidable. As is well known, electromagnetism possess a U(1) symmetry which we might conferred as the unbroken U(1) symmetry in the present universe. Hence this suggest that monopoles should be present in the universe, though no succesful detections have been made so far. For information on experimental detection on monopoles, readers are referred to the reports by Giacomelli (2000) and for more complete mathematical details on monopoles, reports by Goddard and Olive (1978) and Rossi (1982) are referred.

The Georgi-Glashow model proposed by Georgi and Glashow (1972) is believed to be a good toy model for the more realistic GUT models as it possess many properties close to the GUT model. Specifically, this model is known as the SU(2) Yang-Mills-Higgs theory and it contains a gauge field strength tensor $F^a_{\mu\nu}$ coupled with a non-vanishing Higgs field triplet Φ^a which spontaneously breaks the symmetry. We now explore the 't Hooft-Polyakov solution more technically. For more informations on the mathematics of the model readers are referred to chapter four. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a - \frac{\mu^2}{\lambda}\right)^2, \tag{2.1}$$

with the vector gauge fields A^a_μ manifest as the gauge field strength tensor $F^a_{\mu\nu}$, μ is the mass of the Higgs field, and λ is the strength of the Higgs potential. Both μ and λ are constants and the vacuum expectation value of the Higgs field is $\mu/\sqrt{\lambda}$. The Lagrangian (2.1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor $F^a_{\mu\nu}$ are given by

$$D_\mu\Phi^a = \partial_\mu\Phi^a + e\epsilon^{abc}A^b_\mu\Phi^c, \tag{2.2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu. \tag{2.3}$$

where e is the the gauge field coupling constant. By setting e to one here, the equations of motion that follow from the Lagrangian (2.1) are

$$D^\mu F^a_{\mu\nu} = \epsilon^{abc}\Phi^b D_\nu\Phi^c, \quad D^\mu D_\mu\Phi^a = \lambda\Phi^a\left(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}\right). \tag{2.4}$$

The symmetric stress-energy tensor $T_{\mu\nu}$ which follow from the Lagrangian (2.1) and the field equation (2.4) is

$$T_{\mu\nu} = F^a_{\mu\rho}F^{a\rho}_\nu + D_\mu\Phi^a D_\nu\Phi^a - g_{\mu\nu}L. \tag{2.5}$$

From Eq.(2.5) we can easily obtain the static energy as

$$\begin{aligned} E &= \int d^3x\, T_{00} \\ &= \int d^3x\, \frac{1}{2}\left(E^a_nE^a_n + B^a_nB^a_n + (D_n\Phi^a)(D_n\Phi^a)\right) + \frac{\lambda}{4}\left(\Phi^a\Phi^a - \frac{\mu^2}{\lambda}\right)^2 \end{aligned} \tag{2.6}$$

Before we go any further, we would like to stress that solutions of the classical field equations map the vacuum manifold $M = S^2_{\text{vacuum}}$ onto the boundary of 3-dimensional space, which is also a sphere S^2. These maps are characterized by a winding number $n = 0, \pm1, \pm2...$ which is the number of times $M = S^2_{\text{vacuum}}$ is covered by a single turn around the spatial boundary S^2. The important point is that the solutions possessing a finite energy on the spatial asymptotic could be separated into different classes according to the behavior of the field Φ^a. The trivial case is that the isotopic orientation of the fields do not depend on the spatial coordinates and asymptotically the scalar fields tends to the limit

$$\Phi^a = (0, 0, a). \tag{2.7}$$

This situation corresponds to winding number $n = 0$.

There are another type of solutions with the property that the direction of isovector and isoscalar fields in isospace are functions of the spatial coordinates. This is exactly the case 't Hooft (1974) and Polyakov (1974, 1975) considered. To construct the solutions corresponding to the non-trivial of the minimum of the energy functional (2.6), the scalar field on the spatial asymptotic $r \to \infty$ now takes values on the vacuum manifold $|\Phi| = a$. However, the isovector of the scalar field now is directed in the isotopic space along the direction of the radius vector on the spatial asymptotic

$$\Phi^a \to \frac{ar^a}{r}, \tag{2.8}$$

For example, Eq. (2.8) describes a field which, in the x direction in space, has only an isospin '1' component, and has only an isospin '2' and '3' component in the y and z direction respectively in space. In other words, it is 'radial' and Polyakov calls it a 'hedgehog' solution. This asymptotic behavior defines a single mapping of the vacuum M onto the spatial asymptotic, a single turn around the boundary S^2 leads to a single closed path on the sphere $M = S^2_{\text{vacuum}}$ and the winding number of such a mapping is $n = 1$.

For the construction of the monopole solutions, 't Hooft (1974) considered the ansatz

$$A_0^a = 0,\ A_i^a = \epsilon_{amn}\frac{r^m}{er^2}[1-K(r)],\ \Phi^a = H(r)\ \frac{r^a}{er^2}, \tag{2.9}$$

which simplifies the equations of Eq.(2.4) to

$$r^2H'' = 2KH^2 + \frac{\lambda}{e^2}H(H^2-r^2),\quad r^2K'' = KH^2 + K(K^2-1), \tag{2.10}$$

where prime means differentiation with respect to r. To avoid singularity at the origin and achieve non-trivial spatial asymptotic conditions (2.7), the functions K and H obviously must satisfy the following boundary conditions:

$$\begin{aligned} &K(r)\to 1, H(r)\to 0,\quad r\to 0;\\ &K(r)\to 0, H(r)\to r,\quad r\to\infty. \end{aligned} \tag{2.11}$$

Unfortunately, the system of non-linear coupled differential equations (2.10) in general has no analytical solution. The only known exception is the very special case $\lambda = 0$. This is called the Bogomol'nyi-Prasad-Sommerfield (BPS) limit which will be discussed in the next section. However, equations (2.10) can be solved numerically (Bais and Primark, 1976; Kirkman and Zachos, 1981) based on the boundary conditions in Eq.(2.11).

To precisely interpret the solution as a monopole, 't Hooft's approach was to search for a suitable definition of the electromagnetic field within the theory (2.1) and he proposed a tensor that can be identified with the Abelian electromagnetic field tensor,

$$F_{\mu\nu} = \hat{\Phi}^a F_{\mu\nu}^a - \epsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c, \tag{2.12}$$

where $\hat{\Phi}^a$ is the unit vector of the Higgs field. The tensor (2.12) can also be written in a more transparent form (Arafune *et al.*, 1975),

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \tag{2.13}$$

where $A_\mu = \hat{\Phi}^a A^a_\mu, \quad \hat{\Phi}^a = \Phi^a/|\Phi|, \quad |\Phi| = \sqrt{\Phi^a\Phi^a}$. Here A_μ is the massless component of the gauge potential A^a_i. Hence the Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$.

For the ansatz in Eq.(2.9), one easily verifies that $A_\mu = 0$ (in the no-string gauge the massless potential is identically zero). Also $\hat{\Phi}^a = \hat{r}^a$, so

$$F_{0i} = 0, F_{ij} = -(1/e)\varepsilon_{ijk}\hat{r}_k/r^2, \tag{2.14}$$

This is the electromagnetic field of a point magnetic monopole at rest with magnetic charge $g = 1/e$. This corresponds to a radial magnetic field. According to 't Hooft's definition, the electromagnetic field tensor depends only on the unit vector of the Higgs field $\hat{\Phi}^a$. In the string gauge, things are reversed. There $\hat{\Phi}^a = \delta_{a3}$ and the Higgs field terms in Eq.(2.13) vanishes. The massless component of the gauge potential is $A_i = A^3_i$ and $F_{ij} = \partial_i A^3_j - \partial_j A^3_i$.

Numerical results shows that the functions H and K approach rather fast to the asymptotic values (Bais and Primark, 1976; Kirkman and Zachos, 1981). Thus there is a Higgs vacuum outside a finite region, which is of the order of the characteristic scale R_c. This scale is called the core of the monopole and the size of monopole can be estimated by simple arguments, as shown by Preskill (1984). The total energy of the monopole configuration consists of two parts: the energy of the Abelian magnetic field outside the core and the energy of the scalar Higgs field inside the core (Shnir, 2004). Inside the core, the original SU(2) symmetry is restored, but outside the core this symmetry is spontaneously broken down to the Abelian electromagnetic subgroup.

Hence the important properties of the 't Hooft-Polyakov solution are: (1) the gauge potential and Higgs field are nowhere singular, (2) the long range component in the solution correspond to the electromagnetic field of a static magnetic monopole, and (3) the solution has finite energy and is believed to be stable. The stability of these solitons arises from the fact that the boundary conditions fall into distinct classess, of which the vacuum belongs only to one. These boundary conditions are characterized by a particular correspondence (mapping) between the group space and coordinate space. They are topologically distinct because these mappings are not continuously deformable into one another. One might argue that such configurations would be unstable since the absolute minimum of the energy corresponds to the trivial vacuum. However, the stability of them will be secured by the topology, if we try to deform the fields continously to the trivial vacuum (2.7), then the energy functional would tend to infinity. In other words, all the different topological sectors are separated by infinite barriers.

2.2 Prasad-Sommerfield Solution

The numerical 't Hooft-Polyakov solution is obtained under finite value of Higgs potential λ. Here we discuss the very special case $\lambda = 0$. In this limit of vanishing Higgs potential, the Higgs field becomes massless and is not self-interacting. This limit is widely known as the Prasad-Sommerfield (PS) limit. In the PS limit, exact monopole solutions have been obtained by Prasad and Sommerfield (1975). In fact, within the PS limit, the numerical 't Hooft-Polyakov monopole solution becomes the exact Prasad-Sommerfield solution. These exact solutions can be obtained by solving the second order Euler-Lagrange equations as well as the first order Bogomol'nyi equations. Hence the solutions are sometimes said to satisfy the Bogomol'nyi condition or the Bogomol'nyi-Prasad-Sommerfield (BPS) limit. Exact monopole solutions have been obtained only within the BPS limit (so far).

In the BPS limit of vanishing Higgs potential, the scalar field also becomes massless and the energy (2.6) of the system can be written as

$$E = \mp \int \partial_i (B_i^a \Phi^a)\ d^3x + \int \frac{1}{2} (B_i^a \pm D_i \Phi^a)^2\ d^3x. \tag{2.15}$$

The system is said to satisfy the BPS limit if the following Bogomol'nyi equation is satisfied

$$B_i^a \pm D_i \Phi^a = 0, \tag{2.16}$$

and it can also be seen that the energy of the system is independent from the properties of the gauge field and completely defined by the Higgs field alone

$$E = \mp \int \partial_i (B_i^a \Phi^a)\, d^3x = \frac{4\pi}{e} M \frac{\mu}{\sqrt{\lambda}}. \tag{2.17}$$

This shows that the energies of the BPS solutions are minimally bound. Consideration of higher value of λ will give a value of energies higher than that of the Bogomol'nyi bound.

We now look for the exact Prasad-Sommerfield solution. Consideration of the case $\lambda = 0$ simplifies equations (2.10) into

$$r^2 H'' = 2KH^2, \quad r^2 K'' = KH^2 + K(K^2 - 1). \tag{2.18}$$

However we can also obtain the same solution from the first order Bogomol'nyi equation (2.16). Substituting 't Hooft's ansatz into equation (2.16) yields

$$r\frac{dK}{dr} = -KH, \quad r\frac{dH}{dr} = H + (1 - K^2), \tag{2.19}$$

which have an analytical solution in terms of elementary functions:

$$K = \frac{r}{\sinh r}, \quad H = r \coth r - 1, \tag{2.20}$$

Equation (2.20) is the so called Prasad-Sommerfield solution. The solution to the first order BPS equation (2.19) will, of course, automatically satisfies the system of field equations (2.10) of the second order.

In comparison with the 't Hooft-Polyakov solution, the behavior of the Higgs field of the monopole in the BPS limit differs in a dramatic way. The reason for this is that in the limit $V(\phi) = 0$ the scalar field becomes massless and the attractive force associated with the Higgs field becomes long range. Hence the picture of the interaction between the monopoles is quite different in the BPS limit, as compared to the naive picture based on pure electromagnetic interaction. This argument is used in the construction of multimonopole, and will be discussed in the next section.

2.3 Multimonopole

After obtaining the single charged magnetic monopole solutions, it was only by natural desire that one continues to find multimonopole solutions. Besides ensuring the possibility of 't Hooft-Polyakov monopoles to exist together, the multimonopole solutions would also enable one to determine the interaction between 't Hooft-Polyakov monopoles precisely. However, it has been shown by Bogomol'nyi (1976) that there are no spherically symmetric multimonopoles and the $n = 1$ 't Hooft and Polyakov monopole solution is the unique spherically symmetric solution. Weinberg and Guth (1976) also showed that finite energy multimonopole solutions cannot be spherically symmetric but can have at most axial symmetry.

The first step is to investigate the interaction between widely separated singly charged monopoles. Stated in the last sections, it was shown in the BPS limit of vanishing λ, like-charged monopoles are non-interacting (Manton, 1977; Weinberg, 1979; Goldberg *et al.*, 1978; O'Raifeartaigh *et al.*, 1979). This makes the interpretation of non-interacting monopoles evident and the reason behind this is quite simple, the Higgs field mediates an attraction (Manton, 1977) independently of the sign of the magnetic charges, while the Coulomb force due to the unbroken U(1) symmetry is long range. For like-charged monopoles, the Coulomb force is repelling. In the limit of vanishing Higgs potential, the Higgs field becomes massless and the sef-interaction vanishes. The attractive Higgs force becomes long range and exactly cancels the repulsive Coulomb force between like magnetic charges. It is also clear that for $\lambda \neq 0$, the mass of the Higgs field now decays exponentially with distance. At large distance, there are only a repulsive phase and liked charged monopoles should be repelling. This was confirmed numerically by Kleihaus *et al.* (1998).

Hence, in the absence of a repulsive force, nothing prevents in principle the existence of multimonopole solution, and it is possible for the same magnetic charge to get superimposed into one point in space. This was indeed the case as the first exact multimonopole solution was obtained by Ward (1981). It was an exact axially symmetric monopole of topological charge two, with the magnetic charges all superimposed at one point location. Shortly after Wards's work, a generalization to axially symmetric multimonopole solutions with arbitrary topological charge is obtained (Rebbi and Rossi, 1980; Forgacs *et al.*, 1981a; Forgacs *et al.*, 1981b; Prasad, 1981; Prasad and Rossi, 1981). In the BPS limit, these solutions satisfy the Bogolmol'nyi lower bound for all n. The energy per winding number is equal to that of the singly charged monopole, which clealy shows the non-interactions of the monopoles. For the case $\lambda \neq 0$, as a results of the repulsion between like-monopoles, the mass of the n-monopole is always greater than n-times the mass of a single monopole (Kleihaus *et al.*, 1998).

In addition to axially symmetric multimonopoles, multimonopole solutions with discrete symmetries (Sutcliffe, 1997; Houghton *et al.*, 1998) were also found in the limit of $\lambda = 0$ and $n \geq 3$. These solutions were inspired by the observation that Skyrmions have these kind of crystal symmetries (Battye and Sutcliffe, 1997) and were constructed using twistor method. There were also some results which represents multimonopole with finite separation. Brown *et al.* (1982) successfully obtained a two arbitrarily separated SU(2) Yang-Mills-Higgs monopoles by using the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) tecnique. The two zeros of the Higgs field correspond to the location of the two monopoles.

2.4 Dyons

In 1975, using the same model as 't Hooft and Polyakov, Julia and Zee (1975) extended the monopole study by further constructing a dyon solution. This is a classical solution with both magnetic and electric charge. The way to do this is to change the ansatz (2.9) by allowing A_0^a to be non-zero:

$$A_0^a = F(r)\,\frac{r^a}{er^2},\ A_i^a = \epsilon_{amn}\frac{r^m}{er^2}[1-K(r)],\ \Phi^a = H(r)\,\frac{r^a}{er^2}. \tag{2.21}$$

The equations of motion (2.4) then become

$$\begin{aligned} r^2F'' &= 2FH^2, \\ r^2H'' &= 2KH^2 + \frac{\lambda}{e^2}H(H^2-r^2), \\ r^2K'' &= KH^2 + K(K^2-1) - KF^2. \end{aligned} \tag{2.22}$$

A non-zero A_0^a will, of course, give a non-zero F_{0i}^a and therefore an electric field in addition to the magnetic field of the monopole. This extension of 't Hooft-Polyakov solution only becomes meaningful if one can show that a solution exist with $F \neq 0$. Indeed, in the limit of $\mu^2 = 0, \lambda = 0$ with μ^2/λ finite, an explicit solution is known (Prasad and Sommerfield, 1975; Bogomol'nyi, 1976),

$$\begin{aligned} F &= \sinh\gamma(-1+\beta r\frac{\cosh\beta r}{\sinh\beta r}), \\ K &= \frac{\beta r}{\sinh\beta r}, \\ H &= \cosh\gamma(-1+\beta r\frac{\cosh\beta r}{\sinh\beta r}), \end{aligned} \tag{2.23}$$

where β and γ are arbitrary constants. Moreover, when Prasad and Sommerfield (1975) constructed the exact dyon solution (2.23), it became obvious that in the BPS limit the A_0^a component of the gauge field enters the Lagrangian in a similar way as the Higgs field. For non-zero μ^2 and λ, as in the case of the monopole, one cannot solve (2.22) in closed form. However Eq.(2.22) can be solved numerically which satisfy the boundary conditions at infinity,

$$F(r) \to Mr + C_1,\ K(r) \to 0,\ H(r) \to r, \quad r \to \infty, \tag{2.24}$$

where C_1 is the unknown constant that has to be found numerically. The boundary conditions at $r \to 0$ are the same as the monopole case with the profile functions $F(r)$ and $H(r)$ behaving similarly.

To determine the electric charge of the dyon one needs to find the electric field. At large r all definitions of the electromagnetic field tensor are the same and the simple definition $F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu}$ can be used. The dyon electric field at large r is then

$$E_n = \hat{\Phi}^a F^a_{0n} = C_1 r_n / e r^3; \tag{2.25}$$

The dyon electric charge Q can then be calculated by using the Gauss law,

$$Q = (4\pi/e) C_1. \tag{2.26}$$

While the magnetic charge is quantized because of the topological properties of the monopole solution, there is no indication that the electric charge is quantized at the classical level. It is associated with the long range behavior of the now non-vanishing time component A^a_0 of the gauge field at spatial infinity, as shown in Eqs.(2.24) and (2.26). The exact solution (2.23) is also easily shown to be stable. One just needs to consider E^a_n in Eq.(2.6) to be non-zero and repeating the same calculation as in the monopole case gives the energy

$$E = (\mu/\sqrt{\lambda})\sqrt{(g^2 + q^2)}. \tag{2.27}$$

where

$$g = Q = (4\pi/e) C_1 = -(4\pi/e) \sinh\gamma. \tag{2.28}$$

The separate conservation of the electric and magnetic charge implies that this static solution is stable.

From the field equations, Julia and Zee (1975) also emphasized on the condition where $|A^a_0| < 1$ for $r \to \infty$ because if it becomes bigger than one,